Divide & Conquer

- Paradigm
- Convex Hull
- Median finding

Paradigm

Given a problem of size n

Divide it into a subproblems of size $\frac{n}{b}$
$a \geq 1$, $b > 1$

Solve each subproblem recursively

Combine solutions of subproblems to get overall solution

$$T(n) = a\,T\left(\frac{n}{b}\right) + [\text{work for merge}]$$

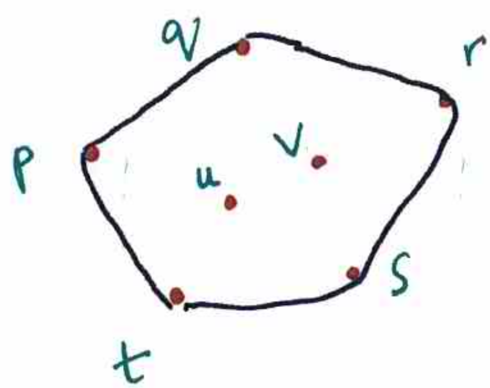
Convex Hull

Given n points in plane [Ref § 33.3]

$$S = \{ (x_i, y_i) \mid i = 1, 2, \ldots n \}$$

assume no two have same x coord, no two have same y coord, and no three in a line for convenience

Convex Hull : smallest polygon containing all points in S
$CH(S)$



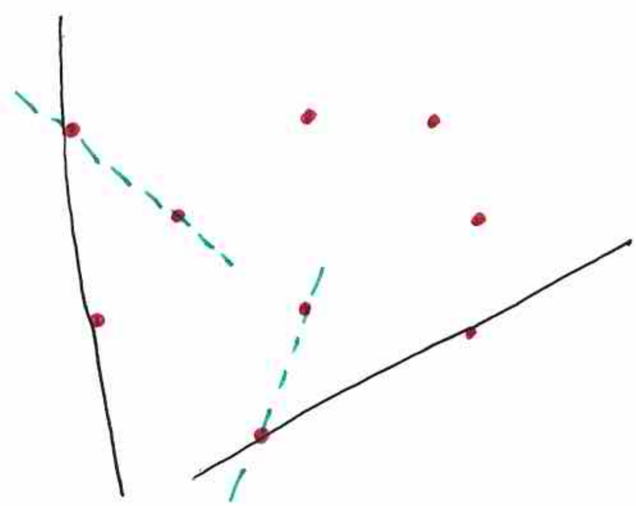
If points are nails, then $CH(S)$ is shape of rubber band around all the nails

$CH(S)$ represented by the sequence of points on the boundary in order $\boxed{\text{clockwise}}$ as doubly linked list

$$p \leftrightarrow q \leftrightarrow r \leftrightarrow s \leftrightarrow t$$

Brute force for Convex Hull

n points



Test each line segment to see if it makes up an edge of the convex hull

→ If the rest of the points are on one side of the segment, the segment is on the convex hull — above

→ else the segment is not - - - above

$O(n^2)$ edges, $O(n)$ tests $\Rightarrow$ $O(n^3)$ complexity

Can we do better?

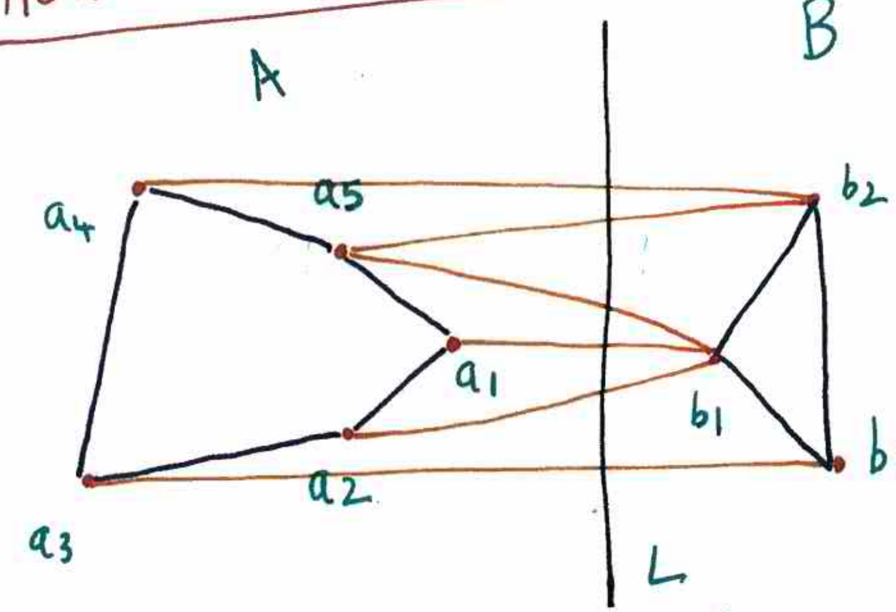
D&C for Convex Hull

Sort points by x coord (once & for all, $O(n \log n)$)

For input set S of points:

- Divide into left-half A & right half B
- by x coords
- Compute $CH(A)$ & $CH(B)$
- Combine CH's of two halves (merge step)

HOW TO MERGE?



(a_4, b_2, b_3, a_3)

merge

Find upper tangent (a_i, b_j) (a_4, b_2) U.T.

Find lower tangent (a_k, b_m) (a_3, b_3) L.T.

Cut & paste in time $\theta(n)$ $(a_1, a_2, a_3, a_4, a_5)$ (b_1, b_2, b_3)

First Link a_i to b_j, go down b list till you see b_m and link b_m to a_k

Continue along the a list until you return to a_i

FINDING TANGENTS

Assume a_1 maximizes x within $CH(A)$ $(a_1, a_2, \ldots a_p)$
b_1 minimizes x within $CH(B)$ $(b_1, b_2, \ldots b_q)$
L is the vertical line separating A & B
Define $y(i,j)$ as y-coordinate of pt of intersection between L & segment (a_i, b_j)

CLAIM: (a_i, b_j) is uppertangent iff it maximizes $y(i,j)$
If $y(i,j)$ is not maximum, there will be points on both sides of (a_i, b_j) and it can't be a tangent.

Algorithm: Obvious $O(n^2)$ algorithm looks at all a_i, b_j pairs $T(n) = 2T(n/2) + \Theta(n^2) = \Theta(n^2)$

$\Theta(n)$

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i = 1
j = 1
while (y(i, j+1) > y(i,j) or y(i-1,j) > y(i,j)):
    if y(i, j+1) > y(i,j):          move right finger ↑
        j = j+1 (mod q)
    else:  i = i-1 (mod p)          move left finger ↑
return (a_i, b_j) as upper tangent
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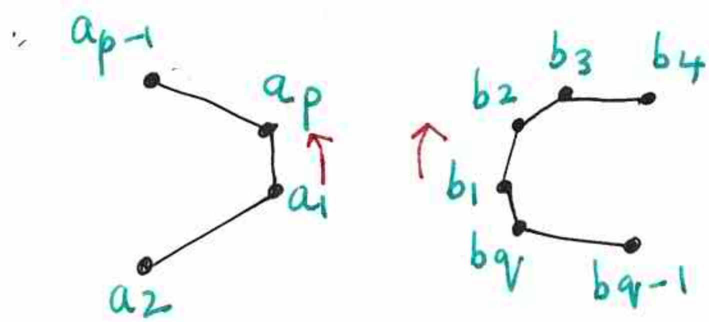
Similarly for lower tangent
$T(n) = 2T(\frac{n}{2}) + \Theta(n)$ Master Theorem gives $\Theta(n \log n)$

Intuition for why Merge works



a_1, b_1 are right most & leftmost points.

We move anti clockwise from a_1,
clockwise from b_1.

$a_1, \ldots a_q$ is a convex hull, as is $b_1, b_2, \ldots b_q$

If a_i, b_j is such that moving from either a_i or b_j decreases $y(i,j)$ there are no points above the (a_i, b_j) line.

The formal proof is quite involved and won't be covered.

Median Finding

[Ref: §9.3]

Given set of n numbers, define $\text{rank}(x)$ as number of numbers in the set that are $\le x$

Find element of rank $\left\lfloor \frac{n+1}{2} \right\rfloor$: lower median

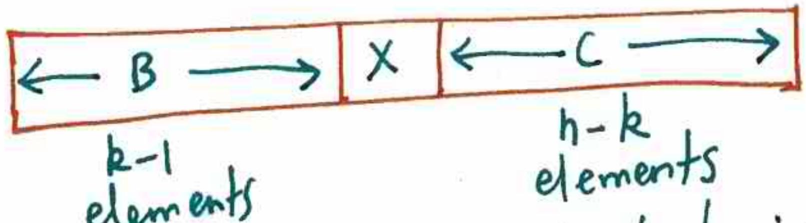
(or element of rank i) $\left\lceil \frac{n+1}{2} \right\rceil$: upper median

Clearly sorting works in time $\theta(n \log n)$

Can we do better?

Select(S, i) (cleverly)

- Pick $x \in S$
- Compute $k = \text{rank}(x)$
  $B = \{y \in S \mid y < x\}$
  $C = \{y \in S \mid y > x\}$

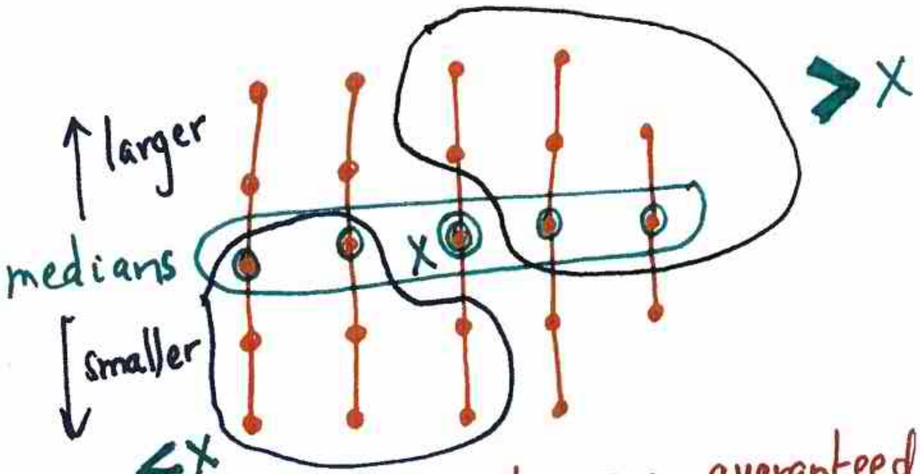


- if $k = i$: return x else if $k > i$: return Select(B, i)
  else if $k < i$: return Select$(C, i-k)$

PICKING X CLEVERLY

Need to pick x so rank(x) is not extreme.

- Arrange S into columns of size 5 ($\lceil \frac{n}{5} \rceil$ cols)
- Sort each column (big elements on top) (linear time)
- Find "median of medians" as X



How many elements are guaranteed to be > X?

Half of the $\lceil n/5 \rceil$ groups contribute at least 3 elements > X except for 1 group with less than 5 elements & 1 group that contains X

At least $3\left(\lceil n/10 \rceil - 2\right)$ elements are > X

" " " " < X

Recurrence: $T(n) = \begin{cases} O(1) & \text{for } n \le 140 \\ \underbrace{T\left(\lceil n/5 \rceil\right)}_{\text{median of medians}} + \underbrace{T\left(\frac{7n}{10} + 6\right)}_{\text{discard } \frac{3n}{10} - 6 \text{ elements}} + \underbrace{\Theta(n)}_{\text{sorting each column}} \end{cases}$

Solving the Recurrence

Master theorem does not apply

Prove $T(n) \leq c \cdot n$ by induction, for some large enough c

INTUITION: $\frac{n}{5} + \frac{7n}{10} < n$

- True for $n \leq 140$ by choosing large c
- $T(n) \leq c \cdot \lceil n/5 \rceil + c\left(\frac{7n}{10} + 6\right) + a \cdot n$

(a needs to be large enough to cover $\theta(n)$ term)

$$\leq \frac{cn}{5} + c + \frac{7nc}{10} + 6c + an$$

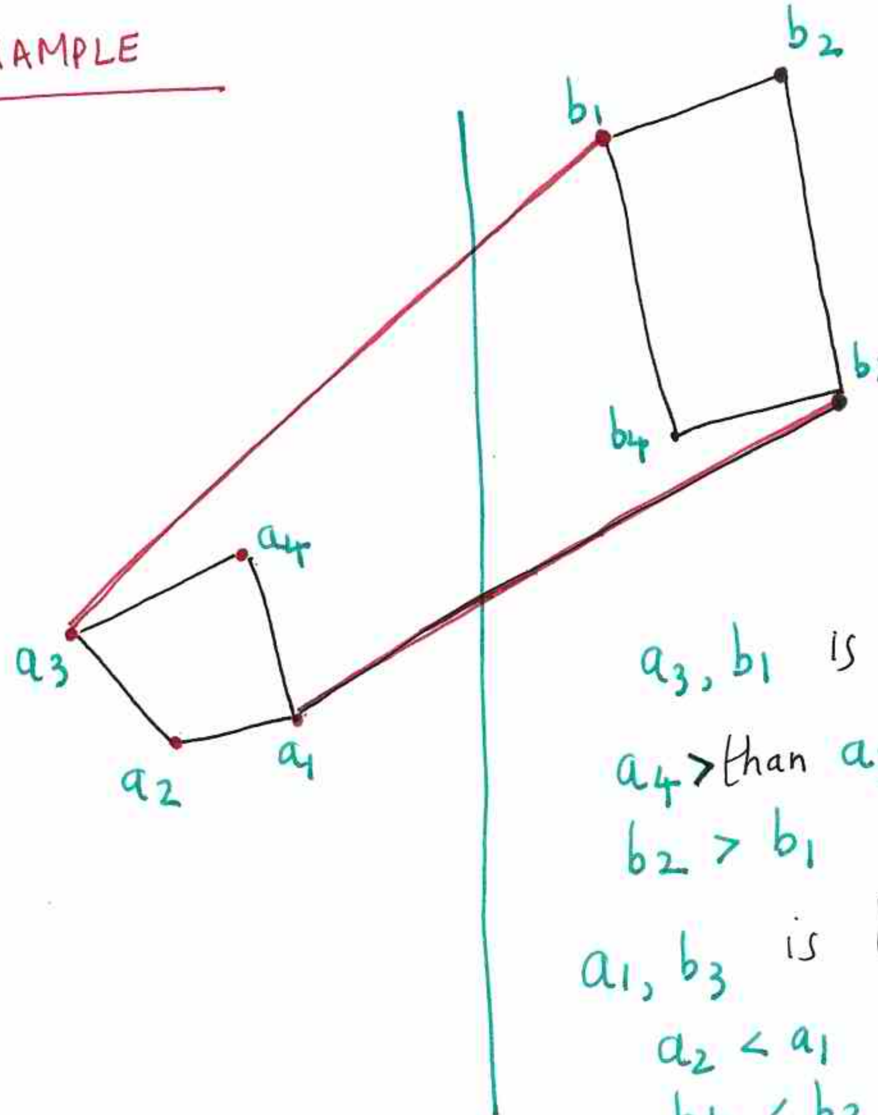
$$= cn + \underbrace{\left(\frac{-cn}{10} + 7c + an\right)}_{\text{if this is } \leq 0 \text{, we are done}}$$

$$c \geq \frac{70c}{n} + 10a$$

ok for $n \geq 140$ & $c \geq 20a$ $\boxtimes$

A

EXAMPLE



a_3, b_1 is upper tangent

$a_4 >$ than a_3

$b_2 > b_1$

a_1, b_3 is lower tangent

$a_2 < a_1$

$b_4 < b_3$

a_i, b_j is an upper tangent. Does not mean that a_i or b_j is the highest point