Integral Boundary Layer Equations

Displacement Thickness

The displacement thickness $\delta^*$ is defined as:

$$
\delta^* = \int_0^\infty \left( 1 - \frac{\rho u}{\rho_u u_e} \right) dy = \int_0^\infty \left( 1 - \frac{u}{u_e} \right) dy
$$

The displacement thickness has at least two useful interpretations:

Interpretation #1

$$A = \int_0^\infty \frac{u}{u_e} dy$$

$$A + B = \int_0^\infty (1) dy$$

So, the difference is in area $B$.

$\Rightarrow \delta^*$ “represents” the decrease in mass flow due to viscous effects, i.e. lost

$$\dot{m}_{visc} = \rho_u u_e \delta^*$$
Interpretation #2

Conservation of mass:

\[
\int_{0}^{y_{1}} u_e \, dy = \int_{0}^{y_{1} + \Delta y} u \, dy \\
\int_{0}^{y_{1}} u_e \, dy = \int_{0}^{y_{1}} u \, dy + \Delta u_e
\]

\[\Rightarrow \Delta u_e = \int_{0}^{y_{1}} (u_e - u) \, dy\]

\[\Delta y = \int_{0}^{y_{1}} \left(1 - \frac{u}{u_e}\right) \, dy\]

Taking the limit of \( y_{1} \to \infty \) gives

\[\Rightarrow \Delta y = \delta^* = \int_{0}^{\infty} \left(1 - \frac{u}{u_e}\right) \, dy\]

So, the external streamline is displaced by a distance \( \delta^* \) away from the body due to viscous effects.

\[\Rightarrow \text{Outer flow sees an “effective body”}\]
Karman’s Integral Momentum Equation

This approach due to Karman leads to a useful approximate solution technique for boundary layer effects. It forms the basis of the boundary layer methods utilized in Prof. Drela’s XFOIL code.

Basic idea: integrate b.l. equations in $y$ to reduce to an ODE in $x$.

Derivation:

Add $(\rho u)$ continuity + $x$-momentum

$$
\Rightarrow \rho u \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho u_e \frac{du_e}{dx} + \mu \frac{\partial^2 u}{\partial y^2}
$$

$$
\Rightarrow \rho \left( \frac{\partial (u^2)}{\partial x} + \frac{\partial}{\partial y} (uv) \right) = \rho u_e \frac{du_e}{dx} + \frac{\partial}{\partial y} \left( \frac{\mu}{\tau} \frac{\partial u}{\partial y} \right)
$$

Now, we integrate from 0 to $y_i$:

$$
\rho \int_0^{y_i} \frac{\partial (u^2)}{\partial x} \, dy + \rho u v|_0^{y_i} = \rho u_e \frac{du_e}{dx} y_i + \tau|_0^{y_i}
$$

Note:

$$
\rho u v|_0^{y_i} = \rho u_e v(y_i) = \rho u_e \int_0^{y_i} \frac{\partial v}{\partial y} \, dy = -\rho u_e \int_0^{y_i} \frac{\partial u}{\partial x} \, dy
$$

So, the equation becomes:
\[
\rho \int_0^{y_1} \frac{\partial (u^2)}{\partial x} \, dy - \rho u \int_0^{y_1} \frac{\partial u}{\partial x} \, dy = \rho u_e \frac{du_e}{dx} y_1 \bigg|_0^{y_1} + \tau \bigg|_0^{y_1}
\]

After a little more manipulation this can be turned into (note we let \( y_1 \to \infty \) also):

\[
\tau_w = \frac{d}{dx} \left( \rho u_e^2 \theta \right) + \rho u_e \delta^* \frac{du_e}{dx}
\]

where \( \theta \equiv \text{momentum thickness} = \int_0^y \frac{\rho u}{\rho u_e} \left( 1 - \frac{\rho u}{\rho u_e} \right) \, dy \)

incompressible form = \[\int_0^y \frac{u}{u_e} \left( 1 - \frac{u}{u_e} \right) \, dy\]

**Insight**

Integrate (1) from stagnation point along airfoil & then down the wake

\[
\int_0^\infty \tau_w \, dx = (\rho u_e^2 \theta) \bigg|_0^\infty + \int_0^\infty \rho u_e \delta^* \frac{du_e}{dx} \, dx
\]

But: \( u_e = 0 \) at stag. pt. \((x = 0)\) & \( \frac{dp}{dx} = \rho u_e \frac{du_e}{dx} \) (Bernoulli)

\[
\Rightarrow \rho u_e^2 \theta \bigg|_{x \to \infty} = \int_0^\infty \tau_w \, dx + \int_0^\infty \delta^* \frac{dp}{dx} \, dx
\]

\[D' = \int_0^\infty \tau_w \, dx + \int_0^\infty \delta^* \frac{dp}{dx} \, dx\] (friction drag)
Another common form of the integral momentum equation is derived below:

\[ \tau_w = \frac{d}{dx}(\rho_e u_e^2 \theta) + \rho_e u_e \delta^* \frac{d u_e}{dx} \]

\[ \frac{\tau_w}{\rho_e u_e^2} = \frac{d \theta}{dx} + \frac{\theta}{u_e} (2 + H) \frac{d u_e}{dx} \]

where

\[ H = \frac{\delta^*}{\theta} \text{ known as "shape parameter"} \]