Unit 11
Membrane Analogy (for Torsion)

Readings:
Rivello 8.3, 8.6
T & G 107, 108, 109, 110, 112, 113, 114

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For a number of cross-sections, we cannot find stress functions. However, we can resort to an analogy introduced by Prandtl (1903).

Consider a **membrane** under pressure $p_i$

*"Membrane": structure whose thickness is small compared to surface dimensions and it (thus) has negligible bending rigidity (e.g. soap bubble)*

$\Rightarrow$ membrane carries load via a constant tensile force along itself.

N.B. Membrane is 2-D analogy of a string  
(plate is 2-D analogy of a beam)

Stretch the membrane over a cutout of the cross-sectional shape in the x-y plane:

*Figure 11.1  Top view of membrane under pressure over cutout*
N = constant tension force per unit length \[ \text{[lbs/in]} \] \[ \text{[N/M]} \]

Look at this from the side:

**Figure 11.2** Side view of membrane under pressure over cutout

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Assume: lateral displacements (w) are small such that no appreciable changes in N occur.

We want to take equilibrium of a small element:

\[
\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}
\]
Figure 11.3  Representation of deformation of infinitesimal element of membrane

Look at side view (one side):

Figure 11.4  Side view of deformation of membrane under pressure

Note: we have similar picture in the x-z plane
We look at equilibrium in the z direction.

Take the z-components of N:

e. g.

\[ z\text{-component} = -N \sin \frac{\partial w}{\partial y} \]

Note +z direction

for small angle:

\[ \sin \frac{\partial w}{\partial y} \approx \frac{\partial w}{\partial y} \]

\[ \Rightarrow z\text{-component} = -N \frac{\partial w}{\partial y} \]

(acts over dx face)
With this established, we get:

\[
\uparrow + \sum F_z = 0 \Rightarrow p_i \, dx \, dy - N \frac{\partial w}{\partial y} \, dx + N \left[ \frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} \, dy \right] \, dx - N \frac{\partial w}{\partial x} \, dy + N \left[ \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \, dx \right] \, dy = 0
\]

Eliminating like terms and canceling out \( dx \, dy \) gives:

\[
p_i + N \frac{\partial^2 w}{\partial y^2} + N \frac{\partial^2 w}{\partial x^2} = 0
\]

\[
\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = - \frac{p_i}{N}
\]

Governing Partial Differential Equation for deflection, \( w \), of a membrane

\textbf{Boundary Condition:} membrane is attached at boundary, so \( w = 0 \) along contour

\[ \Rightarrow \text{Exactly the same as torsion problem:} \]
### Torsion Membrane

**Partial Differential Equation**
\[ \nabla^2 \phi = 2Gk \]

**Boundary Condition**
\[ \phi = 0 \text{ on contour} \]

### Membrane

**Partial Differential Equation**
\[ \nabla^2 w = - \frac{p_i}{N} \]

**Boundary Condition**
\[ w = 0 \text{ on contour} \]

### Analogy:

<table>
<thead>
<tr>
<th>Membrane</th>
<th>Torsion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>( \phi )</td>
</tr>
<tr>
<td>( p_i )</td>
<td>( -k )</td>
</tr>
<tr>
<td>( N )</td>
<td>( \frac{1}{2G} )</td>
</tr>
<tr>
<td>( \frac{\partial w}{\partial x} )</td>
<td>( \frac{\partial \phi}{\partial x} = \sigma_{zy} )</td>
</tr>
<tr>
<td>( \frac{\partial w}{\partial y} )</td>
<td>( \frac{\partial \phi}{\partial y} = -\sigma_{zx} )</td>
</tr>
</tbody>
</table>

**Volume**
\[ \iint w \, dxdy \rightarrow -\frac{T}{2} \]
Note: for orthotropic, would need a membrane to give different N’s in different directions in proportion to $G_{xz}$ and $G_{yz}$

$\Rightarrow$ Membrane analogy only applies to isotropic materials

- This analogy gives a good “physical” picture for $\phi$
- Easy to visualize deflections of membrane for odd shapes

**Figure 11.5**  Representation of $\phi$ and thus deformations for various closed cross-sections under torsion

Can use (and people have used) elaborate soap film equipment and measuring devices

*(See Timoshenko, Ch. 11)*
From this, can see a number of things:

- Location of maximum shear stresses (at the maximum slopes of the membrane)
- Torque applied (volume of membrane)
- “External” corners do not add appreciably to the bending rigidity (J)

⇒ eliminate these:

*Figure 11.6€ Representation of effect of external corners*

⇒ about the same

- Fillets (i.e. @ internal corners) eliminate stress concentrations
To illustrate some of these points let’s consider specifically…
Torsion of a Narrow Rectangular Cross-Section

Figure 11.8  Representation of torsion of structure with narrow rectangular cross-section

Cross-Section

\[ b >> h \]
Use the Membrane Analogy for easy visualization:

**Figure 11.9** Representation of cross-section for membrane analogy

Consider a cross-section in the middle (away from edges):

**Figure 11.10** Side view of membrane under pressure
The governing Partial Differential Equation is:
\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p_i}{N}
\]

Near the middle of the long strip (away from \(y = \pm b/2\)), we would expect \(\frac{\partial^2 w}{\partial y^2}\) to be small. Hence approximate via:
\[
\frac{\partial^2 w}{\partial x^2} \approx -\frac{p_i}{N}
\]

To get \(w\), let’s integrate:
\[
\frac{\partial w}{\partial x} \approx -\frac{p_i}{N}x + C_1
\]
\[
w \approx -\frac{p_i}{2N}x^2 + C_1x + C_2
\]

Now apply the boundary conditions to find the constants:
\[
@ \quad x = + \frac{h}{2}, \quad w = 0
\]
\[ 0 = -\frac{p_i}{2N} \frac{h^2}{4} + C_1 \frac{h}{2} + C_2 \]

\[ @ x = -\frac{h}{2}, \quad w = 0 \]

\[ \Rightarrow 0 = -\frac{p_i}{2N} \frac{h^2}{4} - C_1 \frac{h}{2} + C_2 \]

This gives:

\[ C_1 = 0 \]

\[ C_2 = \frac{p_i h^2}{8N} \]

Thus:

\[ w \approx \frac{p_i}{2N} \left( \frac{h^2}{4} - x^2 \right) \]

Check the volume:

\[ \text{Volume} = \iint w \, dx \, dy \]
integrating over $dy$:

$$h^2 = b \int_{-h/2}^{h/2} p_i \left( \frac{h^2}{4} - x^2 \right) dx$$

$$= p_i b \left[ \frac{h^2}{4} x - \frac{x^3}{3} \right]_{-h/2}^{h/2}$$

$$= \frac{p_i b}{2N} \left[ \frac{h^2 2h}{4 2} - \frac{2 h^3}{3 8} \right]$$

$$\Rightarrow \text{Volume} = \frac{p_i b h^3}{N 12}$$

Using the Membrane Analogy:

$$p_i = -k$$

$$N = \frac{1}{2G}$$

$$\text{Volume} = -\frac{T}{2} = \frac{p_i b h^3}{N 12}$$
\[-\frac{k \cdot b \cdot h^3 \cdot 2G}{12} = -\frac{T}{2}\]

\[\Rightarrow k = -\frac{3T}{G \cdot b \cdot h^3} \quad (k - T \text{ relation})\]

where: \[k = \frac{d\alpha}{dz}\]

So:

\[\frac{d\alpha}{dz} = \frac{T}{GJ}\]

where: \[J = \frac{bh^3}{3}\]

To get the stress:

\[\sigma_{yz} = \frac{\partial w}{\partial x} = -\frac{p_i}{N} x = 2kGx\]

\[\sigma_{yz} = \frac{2T}{J} x \quad (\text{maximum stress is twice that in a circular rod})\]
\[ \sigma_{xz} = \frac{\partial w}{\partial y} = 0 \quad \text{(away from edges)} \]

Near the edges, \( \sigma_{xz} \neq 0 \) and \( \sigma_{yz} \) changes:

\[ \sigma_{yz} = \frac{2T}{J} \chi \quad \text{at these points} \]
(generally, these are the maximum stresses)

Need formulae to correct for “finite” size dependent on ratio \( b/h \). This is the key in \( b \gg h \).
Other Shapes

Through the Membrane Analogy, it can be seen that the previous theory for long, narrow rectangular sections applies also to other shapes.

*Figure 11.12* **Representation of different thin open cross-sectional shapes for which membrane analogy applies**

Slit tube  Channel  I-beam

Consider the above (as well as other similar shapes) as a long, narrow membrane

→ consider the thin channel that then results....
Figure 11.13  Representation of generic thin channel cross-section

![Diagram of a generic thin channel cross-section with dimensions labeled: b, h1, h2, h3, b1, b2, b3.]

Volume \[= \frac{-T}{2}\]

\[
\frac{p_i}{N} \left[ \frac{b_1 h_1^3}{12} + \frac{b_2 h_2^3}{12} + \frac{b_3 h_3^3}{12} \right] = \frac{-T}{2} \quad \text{(from solution for narrow rectangle)}
\]

This gives:
\[-k2G \left[ \frac{b_1 h_1^3}{12} + \frac{b_2 h_2^3}{12} + \frac{b_3 h_3^3}{12} \right] = \frac{-T}{2} \]

\[ \Rightarrow k = \frac{T}{GJ} \quad \Rightarrow \text{k - T relation} \]

where:

\[ J = \frac{1}{3} b_1 h_1^3 + \frac{1}{3} b_2 h_2^3 + \frac{1}{3} b_3 h_3^3 = \sum \frac{1}{3} b_i h_i^3 \]

For the stresses:

\[ \sigma_{yz} = \frac{\partial w}{\partial x} = -\frac{p_i}{N}x = k2Gx = \frac{2T}{J}x \]

\[ \Rightarrow \text{maximum} \quad \left( \text{"local" x} \right) \]

\[ \sigma_{yz} = \frac{2T h_1}{J \ 2} \quad \text{in section } ① \]

\[ \sigma_{yz} = \frac{2T h_2}{J \ 2} \quad \text{in section } ② \]

\[ \sigma_{yz} = \frac{2T h_3}{J \ 2} \quad \text{in section } ③ \]
Figure 11.14  Representation of shear stress “flow” in thin channel under torsion

\[ \sigma_{xz} = \frac{2T}{J} \frac{h_2}{2} \]

Actually have shear concentrations at corners (large slopes \( \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} \))

\[ \Rightarrow \] make “fillets” there

Figure 11.15  Channel cross-section with “fillets” at inner corners
Use the Membrane Analogy for other cross-sections

for example: variable thickness (thin) cross-section

Figure 11.15  **Representation of wing cross-section (variable thickness thin cross-section)***

Using the Membrane Analogy:

\[
J \approx \frac{1}{3} \int_{y_L}^{y_T} h^3 \, dy \quad \quad \quad \sigma_{zy} \approx \frac{2T}{J} \frac{h}{2} \quad \text{etc.}
\]

Now that we’ve looked at open, walled sections; let’s consider closed (hollow) sections. (thick, then thin)