Lecture 2: The Sampling Theorem

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Sampling

• Given a continuous time waveform, can we represent it using discrete samples?
  – How often should we sample?
  – Can we reproduce the original waveform?
The Fourier Transform

- Frequency representation of signals

- Definition:

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \]

\[ x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \]

- Notation:

\[ X(f) = F[x(t)] \]
\[ X(t) = F^{-1} [X(f)] \]
\[ x(t) \leftrightarrow X(f) \]
Unit impulse $\delta(t)$

\[ \delta(t) = 0, \forall t \neq 0 \]

\[ \int_{-\infty}^{\infty} \delta(t) = 1 \]

\[ \int_{-\infty}^{\infty} \delta(t)x(t) = x(0) \]

\[ \int_{-\infty}^{\infty} \delta(t - \tau)x(\tau) = x(t) \]

\[ F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft} dt = e^0 = 1 \]

\[ \delta(t) \leftrightarrow 1 \]
The rectangle pulse function \( \Pi(t) \) is defined as:

\[
\Pi(t) = \begin{cases} 
1 & |t| < 1/2 \\
1/2 & |t| = 1/2 \\
0 & \text{otherwise}
\end{cases}
\]

The Fourier transform of the rectangle pulse function is:

\[
F[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi ft} dt = \int_{-1/2}^{1/2} e^{-j2\pi ft} dt
\]

\[
= \frac{e^{-j\pi f} - e^{j\pi f}}{-j2\pi f} = \frac{\sin(\pi f)}{\pi f} = \text{Sinc}(f)
\]
Properties of the Fourier transform

• Linearity
  - \( x_1(t) \leftrightarrow X_1(f), x_2(t) \leftrightarrow X_2(f) \Rightarrow \alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(f) + \beta X_2(f) \)

• Duality
  - \( X(f) \leftrightarrow x(t) \Rightarrow x(f) \leftrightarrow X(-t) \) and \( x(-f) \leftrightarrow X(t) \)

• Time-shifting: \( x(t-\tau) \leftrightarrow X(f)e^{-j2\pi ft} \)

• Scaling: \( F[(x(at)] = 1/|a| X(f/a) \)

• Convolution: \( x(t) \leftrightarrow X(f), y(t) \leftrightarrow Y(f) \) then,
  - \( F[x(t)*y(t)] = X(f)Y(f) \)
  - Convolution in time corresponds to multiplication in frequency and vice versa

\[
x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\tau)y(\tau) d\tau
\]
Fourier transform properties (Modulation)

\[ x(t) e^{j2\pi f_o t} \iff X(f - f_o) \]

Now, \( \cos(x) = \frac{e^{jx} + e^{-jx}}{2} \)

\[ x(t) \cos(2\pi f_o t) = \frac{x(t)e^{j2\pi f_o t} + x(t)e^{-j2\pi f_o t}}{2} \]

Hence, \( x(t) \cos(2\pi f_o t) \iff X(f - f_o) + X(f + f_o) \)

- Example: \( x(t) = \text{sinc}(t), \quad F[\text{sinc}(t)] = \Pi(f) \)

- \( Y(t) = \text{sinc}(t)\cos(2\pi f_o t) \iff (\Pi(f-f_o) + \Pi(f+f_o))/2 \)
More properties

- **Power content of signal**
  \[
  \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df
  \]

- **Autocorrelation**
  \[
  R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t - \tau) dt
  \]

- **Sampling**
  \[
  x(t_o) = x(t)\delta(t - t_o)
  \]
  \[
  x(t) \sum_{n=\infty}^{\infty} \delta(t - nt_o) = \text{sampled version of } x(t)
  \]
  \[
  F[ \sum_{n=\infty}^{\infty} \delta(t - nt_o)] = \frac{1}{t_o} \sum_{n=\infty}^{\infty} \delta\left(f - \frac{n}{t_o}\right)
  \]
The Sampling Theorem

- Band-limited signal
  - Bandwidth < W

Sampling Theorem: If we sample the signal at intervals $T_s$ where $T_s \leq 1/2W$ then signal can be completely reconstructed from its samples using the formula

$$x(t) = \sum_{n=-\infty}^{\infty} 2W'T_s x(nT_s) \text{sinc}[2W'(t-nT_s)]$$

Where, $W \leq W' \leq \frac{1}{T_s} - W$

$$With \ T_s = \frac{1}{2W} \Rightarrow x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}[\frac{t}{T_s} - n]$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(\frac{n}{2W}) \text{sinc}[2W(t - \frac{n}{2W})]$$
Proof

\[ x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - n T_s) \]

\[ X_\delta(f) = X(f) * F[ \sum_{n=-\infty}^{\infty} \delta(t - n T_s)] \]

\[ F[ \sum_{n=-\infty}^{\infty} \delta(t - n T_s)] = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) \]

\[ X_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s}) \]

- The Fourier transform of the sampled signal is a replication of the Fourier transform of the original separated by 1/Ts intervals
• If $1/T_s > 2W$ then the replicas of $X(f)$ will not overlap and can be recovered.

• How can we reconstruct the original signal?
  – Low pass filter the sampled signal

• Ideal low pass filter is rectangular
  – Its impulse response is a sinc function

• Now the recovered signal after low pass filtering

\[
X(f) = X_\delta(f)T_s \Pi\left(\frac{f}{2W}\right)
\]

\[
x(t) = F^{-1}[X_\delta(f)T_s \Pi\left(\frac{f}{2W}\right)]
\]

\[
x(t) = \sum_{n = -\infty}^{\infty} x(nT_s)Sinc\left(\frac{t}{T_s} - n\right)
\]
Notes about Sampling Theorem

- When sampling at rate 2W the reconstruction filter must be a rectangular pulse
  - Such a filter is not realizable
  - For perfect reconstruction must look at samples in the infinite future and past

- In practice we can sample at a rate somewhat greater than 2W which makes reconstruction filters that are easier to realize

- Given any set of arbitrary sample points that are 1/2W apart, can construct a continuous time signal band-limited to W

- Sampling using “impulses” is also not practical
  - Narrow pulses are difficult to implement
  - In practice, sampling is done using small rectangular pulses or “zero-order-hold”
Zero-Order Hold

- A form of “interpolation”

- The sampled signal holds its value until the next sample time

- In principle, zero-order hold can be realized with a cascade of an impulse train sampling and an LTI system with rectangular impulse response

\[ x(t) \xrightarrow{p(t)} x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s) \xrightarrow{h_0(t)} x_0(t) \]
Reconstruction from zero-order hold

- We know from the sampling theorem that in order to reconstruct $x(t)$ from the impulse train samples on the left ($x_\delta(t)$) the filter on the right ($H(f)$) must be an ideal rectangular filter

$$H(f) = T_s \Pi\left(\frac{f}{2W}\right) = T_s \Pi(T_s f)$$

$$H(f) = T_s \Pi\left(\frac{f}{2W}\right) = H_0(f)H_r(f)$$

$$H_0(f) = e^{-j\pi f T} \left(\frac{\sin(\pi f T)}{\pi f}\right)$$

$$\Rightarrow H_r(f) = T_s \Pi\left(\frac{f}{2W}\right)e^{j\pi f} \pi f \left(\frac{\sin(\pi f T)}{\sin(\pi f T)}\right)$$
Aliasing

• Sampling theorem requires that the signal be sampled at a frequency greater than twice its bandwidth

• When sampling at a frequency less than 2W, the replicas of the frequency spectrum overlap and cannot be “separated” using a low pass filter

• This is referred to as aliasing
  – Higher frequencies are “reflected” only lower frequencies
  – Signal cannot be recovered

• The term aliasing refers to the fact that the higher frequency signals become indistinguishable from the lower frequency ones