Lectures 8 - 9: Signal Detection in Noise and the Matched Filter

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Noise in communication systems

\[ r(t) = S(t) + n(t) \]

- Noise is additional “unwanted” signal that interferes with the transmitted signal
  - Generated by electronic devices

- The noise is a random process
  - Each “sample” of \( n(t) \) is a random variable

- Typically, the noise process is modeled as “Additive White Gaussian Noise” (AWGN)
  - White: Flat frequency spectrum
  - Gaussian: noise distribution
• The auto-correlation of a random process $x(t)$ is defined as
  \[ R_{xx}(t_1,t_2) = E[x(t_1)x(t_2)] \]

• A random process is Wide-sense-stationary (WSS) if its mean and auto-correlation are not a function of time. That is
  \[ m_x(t) = E[x(t)] = m \]
  \[ R_{xx}(t_1,t_2) = R_x(\tau), \text{ where } \tau = t_1 - t_2 \]

• If $x(t)$ is WSS then:
  \[ R_x(\tau) = R_x(-\tau) \]
  \[ |R_x(\tau)| \leq |R_x(0)| \text{ (max is achieved at } \tau = 0) \]

• The power content of a WSS process is:
  \[ P_x = E[ \lim_{t \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt ] = \lim_{t \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} R_x(0)dt = R_x(0) \]
Power Spectrum of a random process

- If $x(t)$ is WSS then the power spectral density function is given by:
  \[
  S_x(f) = \mathcal{F}[R_x(\tau)]
  \]

- The total power in the process is also given by:

\[
\begin{align*}
P_x &= \int_{-\infty}^{\infty} S_x(f) df = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} dt \right] df \\
&= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} df \right] dt \\
&= \int_{-\infty}^{\infty} R_x(t) \left[ \int_{-\infty}^{\infty} e^{-j2\pi ft} df \right] dt = \int_{-\infty}^{\infty} R_x(t) \delta(t) dt = R_x(0)
\end{align*}
\]
White noise

- The noise spectrum is flat over all relevant frequencies
  - White light contains all frequencies

- Notice that the total power over the entire frequency range is infinite
  - But in practice we only care about the noise content within the signal bandwidth, as the rest can be filtered out

- After filtering the only remaining noise power is that contained within the filter bandwidth (B)
**AWGN**

- The effective noise content of bandpass noise is $BN_o$
  - Experimental measurements show that the pdf of the noise samples can be modeled as zero mean gaussian random variable

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2 / 2\sigma^2}$$

  - AKA Normal r.v., $N(0,\sigma^2)$
  - $\sigma^2 = P_X = BN_o$

- The CDF of a Gaussian R.V.,

$$F_X(\alpha) = P[X \leq \alpha] = \int_{-\infty}^{\alpha} f_X(x)dx = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2 / 2\sigma^2} dx$$

- This integral requires numerical evaluation
  - Available in tables
AWGN, continued

- $X(t) \sim N(0, \sigma^2)$
- $X(t_1), X(t_2)$ are independent unless $t_1 = t_2$

$$R_x(\tau) = E[X(t + \tau)X(t)] = \begin{cases} E[X(t + \tau)]E[X(t)] & \tau \neq 0 \\ E[X^2(t)] & \tau = 0 \end{cases}$$

$$= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 & \tau = 0 \end{cases}$$

- $R_x(0) = \sigma^2 = P_x = B\!N_0$
Detection of signals in AWGN

Observe: \( r(t) = S(t) + n(t), \ t \in [0,T] \)

Decide which of \( S_1, \ldots, S_m \) was sent

- **Receiver filter**
  - Designed to maximize signal-to-noise power ratio (SNR)

![Diagram](image)

- **Goal:** find \( h(t) \) that maximized SNR
**Receiver filter**

\[
y(t) = r(t) * h(t) = \int_0^t r(\tau)h(t - \tau) d\tau
\]

**Sampling at** \( t = T \) \( \Rightarrow \)

\[
y(T) = \int_0^T r(\tau)h(T - \tau) d\tau
\]

\[
r(\tau) = s(\tau) + n(\tau) \Rightarrow
\]

\[
y(T) = \int_0^T s(\tau)h(T - \tau) d\tau + \int_0^T n(\tau)h(T - \tau) d\tau = Y_s(T) + Y_n(T)
\]

\[
SNR = \frac{Y_s^2(T)}{E[Y_n^2(T)]} = \frac{\left[ \int_0^T s(\tau)h(T - \tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t) dt}
\]

\[
= \frac{\left[ \int_0^T h(\tau)s(T - \tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t) dt}
\]
Matched filter: maximizes SNR

Cauchy-Schwartz Inequality:

\[
\left( \int_{-\infty}^{\infty} g_1(t) g_2(t) dt \right)^2 \leq \int_{-\infty}^{\infty} (g_1(t))^2 \int_{-\infty}^{\infty} (g_2(t))^2
\]

Above holds with equality iff: \( g_1(t) = c g_2(t) \) for arbitrary constant \( c \)

\[
SNR = \frac{\left[ \int_{0}^{T} s(\tau) h(T-\tau) d\tau \right]^2}{\frac{N_0}{2} \int_{0}^{T} h^2(T-t) dt} \leq \frac{\left[ \int_{0}^{T} (s(\tau))^2 d\tau \int_{0}^{T} h^2(T-\tau) d\tau \right]}{\frac{N_0}{2} \int_{0}^{T} h^2(T-t) dt} = \frac{2}{N_0} \int_{0}^{T} (s(\tau))^2 d\tau = \frac{2E_s}{N_0}
\]

Above maximum is obtained iff: \( h(T-\tau) = cS(\tau) \)

\( \Rightarrow h(t) = cS(T-t) = S(T-t) \)

\( h(t) \) is said to be “matched” to the signal \( S(t) \)
Example: PAM

\[ S_m(t) = A_m g(t), \quad t \in [0,T] \]

\( A_m \) is a constant: Binary PAM \( A_m \in \{0,1\} \)

Matched filter is matched to \( g(t) \)

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\[ g(t) \]

\[ g(T-t) \]  “matched filter”

---

A \[ T \]  A \[ T \]
Example, continued

\[ Y_s(t) = \int_{0}^{t} S(\tau)h(t - \tau)d\tau, \quad h(t') = g(T - t') \Rightarrow h(t - \tau) = g(T - \tau - t) \]

\[ Y_s(t) = \int_{0}^{t} g(\tau)g(T + \tau - t)d\tau = \int_{0}^{t} g(\tau)g(T - t + \tau)d\tau \]

\[ Y_s(T) = \int_{0}^{T} g^2(\tau)d\tau \]

- Sample at \( t = T \) to obtain maximum value
Matched filter receiver

\[ U(t) \rightarrow r_x(t) \rightarrow g(T-t) \rightarrow r_x(kT) \]

Sample at \( t=kT \)

\[ U(t) \rightarrow r_y(t) \rightarrow g(T-t) \rightarrow r_y(kT) \]

Sample at \( t=kT \)

\[ 2\cos(2\pi f_c t) \]

\[ 2\sin(2\pi f_c t) \]
Binary PAM example, continued

\[ 0 \Rightarrow S_1 = g(t) \]
\[ 1 \Rightarrow S_2 = -g(t) \]

![Diagram showing examples of binary signals S(t) and Y(t)]
Alternative implementation: correlator receiver

\[ r(t) = S(t) + n(t) \]

Sample at \( t = kT \)

\[ Y(T) = \int_0^T r(t)S(t) = \int_0^T S^2(t) + \int_0^T n(t)S(t) = Y_s(T) + Y_n(T) \]

Notice resemblance to matched filter