16.410/413
Principles of Autonomy and Decision Making
Lecture 16: Mathematical Programming I

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Assignments

Readings

- Lecture notes
- [IOR] Chapters 2, 3, 9.1-3.
- [PA] Chapter 6.1-2
## Shortest Path Problems on Graphs

### Input: \( \langle V, E, w, s, G \rangle \):
- **V**: set of vertices (finite, or in some cases countably infinite).
- **E \subseteq V \times V**: set of edges.
- **w : E \rightarrow \mathbb{R}_+**, \( e \mapsto w(e) \): a function that associates to each edge a strictly positive weight (cost, length, time, fuel, prob. of detection).
- **S, G \subseteq V**: respectively, start and end sets. Either \( S \) or \( G \), or both, contain only one element. For a point-to-point problem, both \( S \) and \( G \) contain only one element.

### Output: \( \langle T, W \rangle \)
- **T** is a weighted tree (graph with no cycles) containing one minimum-weight path for each pair of start-goal vertices \((s, g) \in S \times G\).
- **W : S \times G \rightarrow \mathbb{R}_+** is a function that returns, for each pair of start-goal vertices \((s, g) \in S \times G\), the weight \( W(s, g) \) of the minimum-weight path from \( s \) to \( g \). The weight of a path is the sum of the weights of its edges.
Example: point-to-point shortest path

Find the minimum-weight path from $s$ to $g$ in the graph below:

Solution: a simple path $P = \langle s, a, d, g \rangle$ ($P = \langle s, b, d, g \rangle$ would be acceptable, too), and its weight $W(s, g) = 8$. 
Another look at shortest path problems

Cost formulation

- The cost of a path \( P \) is the sum of the cost of the edges on the path. *Can we express this as a simple mathematical formula?*
  - Label all the edges in the graph with consecutive integers, e.g., \( E = \{e_1, e_2, \ldots, e_{n_E}\} \).
  - Define \( w_i = w(e_i) \), for all \( i \in 1, \ldots, n_E \).
  - Associate with each edge a variable \( x_i \), such that:
    \[
    x_i = \begin{cases} 
      1 & \text{if } e_i \in P, \\
      0 & \text{otherwise}.
    \end{cases}
    \]

- Then, the cost of a path can be written as:
  \[
  \text{Cost}(P) = \sum_{i=1}^{n_E} w_i x_i.
  \]

- Notice that the cost is a **linear function** of the unknowns \( \{x_i\} \).
Another look at shortest path problems (2)

**Constraints formulation**

- Clearly, if we just wanted to minimize the cost, we would choose $x_i = 0$, for all $i = 1, \ldots, n_E$: this would not be a path connecting the start and goal vertices (in fact, it is the empty path).

- Add these constraints:
  - There must be an edge in $P$ that goes out of the start vertex.
  - There must be an edge in $P$ that goes into the goal vertex.
  - Every (non start/goal) node with an incoming edge must have an outgoing edge.

- A neater formulation is obtained by adding a “virtual” edge $e_0$ from the goal to the start vertex:
  - $x_0 = 1$, i.e., the virtual edge is always chosen.
  - Every node with an incoming edge must have an outgoing edge.
Another look at shortest path problems (3)

- Summarizing, what we want to do is:

  \[
  \text{minimize} \quad \sum_{i=1}^{n_E} w_i x_i \\
  \text{subject to:} \quad \sum_{e_i \in \text{In}(s)} x_i - \sum_{e_j \in \text{Out}(s)} x_j = 0, \quad \forall s \in V; \\
  x_j \geq 0, \quad i = 1, \ldots, n_E; \\
  x_0 = 1.
  \]

- It turns out that the solution of this problem yields the shortest path. (Interestingly, we do not have to set that \( x_i \in \{0, 1\} \), this will be automatically satisfied by the optimal solution!)
Consider again the following shortest path problem:

\[
\begin{align*}
\min \quad & 2x_1 + 5x_2 + 4x_3 + 2x_4 + x_5 + 5x_6 + 3x_7 + 2x_8 \\
\text{s.t.:} \quad & x_0 - x_1 - x_2 = 0, \text{(node } s) \\
& x_1 - x_3 - x_4 = 0, \text{(node } a) \\
& x_2 - x_5 - x_6 = 0, \text{(node } b) \\
& x_4 - x_7 = 0, \text{(node } c) \\
& x_3 + x_5 + x_7 - x_8 = 0, \text{(node } c) \\
& x_2 + x_5 - x_0 = 0, \text{(node } g) \\
& x_i \geq 0, \quad i = 1, \ldots, 8; \\
& x_0 = 1.
\end{align*}
\]

Notice: cost function and constraints are affine ("linear") functions of the unknowns \((x_1, \ldots, x_8)\).
A fire-fighting problem: formulation

Three fires
- Fire 1 needs 1000 units of water;
- Fire 2 needs 2000 units of water;
- Fire 3 needs 3000 units of water.

Two fire-fighting autonomous aircraft
- Aircraft A can deliver 1 unit of water per unit time;
- Aircraft B can deliver 2 units of water per unit time.

Objective
It is desired to extinguish all the fires in minimum time.
A fire-fighting problem: formulation (2)

- Let $t_{A1}, t_{A2}, t_{A3}$ the time vehicle $A$ devotes to fire 1, 2, 3, respectively.
- Define $t_{B1}, t_{B2}, t_{B3}$ in a similar way, for vehicle $B$.
- Let $T$ be the total time needed to extinguish all three fires.
- Optimal value (and optimal strategy) found solving the following problem:

\[
\begin{align*}
\min \quad & T \\
\text{s.t.:} \quad & t_{A1} + 2t_{B1} = 1000, \\
& t_{A2} + 2t_{B2} = 2000, \\
& t_{A3} + 2t_{B3} = 3000, \\
& t_{A1} + t_{A2} + t_{A3} \leq T, \\
& t_{B1} + t_{B2} + t_{B3} \leq T, \\
& t_{A1}, t_{A2}, t_{A3}, t_{B1}, t_{B2}, t_{B3}, T \geq 0.
\end{align*}
\]

- (if you are curious about the solution, the optimal $T$ is 2000 time units)
Many (most, maybe all?) problems in engineering can be defined as:

- A set of constraints defining all candidate ("feasible") solutions, e.g., \( g(x) \leq 0 \).
- A cost function defining the "quality" of a solution, e.g., \( f(x) \).

The formalization of a problem in these terms is called a **Mathematical Program**, or **Optimization Problem**. *(Notice this has nothing to do with "computer programs!")*

The two problems we just discussed are examples of mathematical program. Furthermore, both of them are such that both \( f \) and \( g \) are affine functions of \( x \). Such problems are called **Linear Programs**.
Mathematical Programming

Linear Programming
- Historical notes
- Geometric Interpretation
- Reduction to standard form

Geometric Interpretation
Linear Programs

- The **Standard Form** of a linear program is an optimization problem of the form

\[
\begin{align*}
\text{max} & \quad z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n, \\
\text{s.t.:} & \quad a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1, \\
& \quad a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2, \\
& \quad \ldots \\
& \quad a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m, \\
& \quad x_1, x_2, \ldots, x_n \geq 0.
\end{align*}
\]

- In a more compact form, the above can be rewritten as:

\[
\begin{align*}
\text{min} & \quad z = c^T x, \\
\text{s.t.:} & \quad Ax = b, \\
& \quad x \geq 0.
\end{align*}
\]
Historical Notes

- Historical contributor: **G. Dantzig** (1914-2005), in the late 1940s. (He was at Stanford University.) Realize many real-world design problems can be formulated as linear programs and solved efficiently. Finds algorithm, the Simplex method, to solve LPs. As of 1997, still best algorithm for most applications.
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- So important for world economy that any new algorithmic development on LPs is likely to make the front page of major newspapers (e.g. NY times, Wall Street Journal). Example: 1979 L. Khachyans adaptation of ellipsoid algorithm, N. Karmarkars new interior-point algorithm.
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A remarkably practical and theoretical framework: LPs eat a large chunk of total scientific computational power expended today. It is crucial for economic success of most distribution/transport industries and to manufacturing.

Now becomes suitable for real-time applications, often as the fundamental tool to solve or approximate much more complex optimization problem.
Geometric Interpretation

- Consider the following simple LP:

  \[
  \text{max} \quad z = x_1 + 2x_2 = (1, 2) \cdot (x_1, x_2),
  \]

  \[
  \text{s.t.:} \quad x_1 \leq 3, \\
  x_1 + x_2 \leq 5, \\
  x_1, x_2 \geq 0.
  \]

- Each inequality constraint defines a hyperplane, and a **feasible** half-space.

- The intersection of all feasible half spaces is called the **feasible region**.

- The feasible region is a (possibly **unbounded**) polyhedron.

- The feasible region could be the empty set: in such case the problem is said **unfeasible**.
Geometric Interpretation (2)

Consider the following simple LP:

$$\text{max } z = x_1 + 2x_2 = (1, 2) \cdot (x_1, x_2),$$

s.t.:

- $x_1 \leq 3$,
- $x_1 + x_2 \leq 5$,
- $x_1, x_2 \geq 0$.

The “c” vector defines the gradient of the cost.

Constant-cost loci are planes normal to $c$.

Most often, the optimal point is located at a vertex (corner) of the feasible region.

- If there is a single optimum, it must be a corner of the feasible region.
- If there are more than one, two of them must be adjacent corners.
- If a corner does not have any adjacent corner that provides a better solution, then that corner is in fact the optimum.
Converting a LP into standard form

- Convert to maximization problem by flipping the sign of c.
- Turn all “technological” inequality constraints into equalities:
  - **less than** constraints: introduce slack variables.
    \[ \sum_{j=1}^{n} a_{ij}x_j \leq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i, \quad s_i \geq 0. \]
  - **greater than** constraints: introduce excess variables.
    \[ \sum_{j=1}^{n} a_{ij}x_j \geq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j - e_i = b_i, \quad e_i \geq 0. \]
- Flip the sign of non-positive variables: \( x_i \leq 0 \Rightarrow x'_i = -x_i \geq 0. \)
- If a variable does not have sign constraints, use the following trick:
  \[ x_i \Rightarrow x'_i - x''_i, \quad x'_i, x''_i \geq 0. \]
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\text{s.t.:} \quad & x_1 \leq 3, \\
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s.t.: \[
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A naïve algorithm (1)

- Recall the standard form:

\[
\begin{align*}
\min & \quad z = c^T x \\
\text{s.t.:} & \quad Ax = b, \\
& \quad x \geq 0.
\end{align*}
\]

- Corners of the feasible regions (also called **basic feasible solutions**) are solutions of \( Ax = b \) (\( m \) equations in \( n \) unknowns, \( n > m \)), obtained setting \( n - m \) variables to zero, and solving for the others (basic variables), ensuring that all variables are non-negative.
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& \quad x \geq 0.
\end{align*}
\]

or, really:

\[
\begin{align*}
\min & \quad z = c^T y + c^T s \\
\text{s.t.:} & \quad A_y y + l s = b, \\
& \quad y, s \geq 0.
\end{align*}
\]

- Corners of the feasible regions (also called basic feasible solutions) are solutions of \(Ax = b\) (\(m\) equations in \(n\) unknowns, \(n > m\)), obtained setting \(n - m\) variables to zero, and solving for the others (basic variables), ensuring that all variables are non-negative.

- This amounts to:
  - picking \(n_y\) inequality constraints, (notice that \(n = n_y + n_s = n_y + m\)).
  - making them active (or binding),
  - finding the (unique) point where all these hyperplanes meet.
  - If all the variables are non-negative, this point is in fact a vertex of the feasible region.
A naïve algorithm (2)

- One could possibly generate all basic feasible solutions, and then check the value of the cost function, finding the optimum by enumeration.

- **Problem**: how many candidates?

\[
\binom{n}{n-m} = \frac{n!}{m!(n-m)!}.
\]

- for a “small” problem with \(n = 10, \ m = 3\), we get 120 candidates.
- this number grows very quickly, the typical size of realistic LPs is such that \(n,m\) are often in the range of several hundreds, or even thousands.

- Much more clever algorithms exist: stay tuned.
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