LECTURE # 12
RIGID BODY DYNAMICS

- IMPLICATIONS OF $\mathbf{\dot{\omega} = \Omega \times} $

- GENERAL ROTATIONAL DYNAMICS
  - EULER'S EQUATION OF MOTION

- TORQUE FREE SPECIAL CASES.

PRIMARY LESSONS:
- 3D ROTATIONAL MOTION MUCH MORE COMPLEX THAN PLANAR (2D)

- EULER'S E.O.M. PROVIDE STARTING POINT FOR ALL A/C + S/C DYNAMICS

- SOLUTIONS TO EULER'S EQUATIONS ARE COMPLEX, BUT WE CAN DEVELOP GOOD GEOMETRIC VISUALIZATION TOOLS.
• Now can develop the full set of rotational dynamics:

\[ \dot{\mathbf{M}} = \dot{\mathbf{H}} = \dot{\mathbf{H}} + \dot{\mathbf{W}} \times \dot{\mathbf{H}} \]

\( \dot{\mathbf{H}} \) denotes body frame.

\( \dot{\mathbf{W}} \) angular velocity of body w.r.t. inertial.

• Now, we assume that we are using a frame for the body that is centered at the center of mass and fixed to the body.

\[ \Rightarrow \dot{\mathbf{I}} \mathbf{b} = 0 \quad \text{inertia values fixed.} \]

\[ \therefore \mathbf{H} = \mathbf{H} = \frac{d}{dt} (\frac{1}{2} \dot{\mathbf{w}}) = \frac{1}{2} \mathbf{I} \cdot \dot{\mathbf{w}} \]

Recall, if \( \dot{\mathbf{w}} = \dot{w}_x \mathbf{i} + \dot{w}_y \mathbf{j} + \dot{w}_z \mathbf{k} \)

then \( \dot{\mathbf{w}} \mathbf{b} = \dot{w}_x \mathbf{i} + \dot{w}_y \mathbf{j} + \dot{w}_z \mathbf{k} \)

• Summary:

\[ \dot{\mathbf{M}} = \dot{\mathbf{H}} = \frac{1}{2} \mathbf{I} \cdot \dot{\mathbf{w}} \mathbf{b} + \dot{\mathbf{w}} \times (\frac{1}{2} \mathbf{I} \cdot \dot{\mathbf{w}}) \]

- General form of rotational dynamics.
IF WE NOW USE THE BODY FRAME, CAN WRITE THESE IN MATRIX FORM:

\[ M_B = I_B \dot{w}_B + \dot{w}_B^T I_B w_B \]

\[ I_B = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yz} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \]

VERY COMPLEX FOR FULL \( I_B \)

\( \Rightarrow \) SIMPLIFIES IF WE ASSUME THAT BODY FRAME ALIGNED WITH PRINCIPAL AXES.

\( \Rightarrow \)

\[ I_B = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \]

REDUCES TO EULER'S EQUATIONS OF MOTION:

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} = \begin{bmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{bmatrix} \begin{bmatrix}
\dot{w}_x \\
\dot{w}_y \\
\dot{w}_z
\end{bmatrix} + \begin{bmatrix}
0 & -w_z & w_y \\
w_z & 0 & -w_x \\
-w_y & w_x & 0
\end{bmatrix} \begin{bmatrix}
I_{xx} \dot{w}_x \\
I_{yy} \dot{w}_y \\
I_{zz} \dot{w}_z
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I_{xx} \dot{w}_x + (I_{zz} - I_{yy}) w_y w_z \\
I_{yy} \dot{w}_y + (I_{xx} - I_{zz}) w_x w_z \\
I_{zz} \dot{w}_z + (I_{yy} - I_{xx}) w_x w_y
\end{bmatrix}
\]
- Euler's Equations
  - Nonlinear, coupled, few analytic solutions.

- Typically two problems of interest:
  1. Given \( \dot{\mathbf{m}} \), what is the response of the system (given a motion, what must \( \ddot{\mathbf{m}} \) be?)
  2. In the absence of \( \dot{\mathbf{m}} \) (torque free) what would the motion of the body be?

1. "Given motion, find \( \dot{\mathbf{m}} \)" is relatively simple. Much harder the other way (given \( \dot{\mathbf{m}} \), find \( \dot{\mathbf{\omega}} \)).
  - Requires solution of the coupled, nonlinear equations - few analytic answers.
  - Example → easily done numerically

2. Can give a lot of geometric insights into what types of motions occur.
  - Momentum + energy ellipsoids.
  → "torque free" motion only.
EXAMPLE: BEER + JOHNSTON 18.67

- SHAYT WEIGHS 16-1G
- ROTATES AT CONSTANT RATE
  \[ W = 9 \text{ RAD/SEC} \]
- FIND REACTIONS AT POINTS A, B.

SOLUTION: - FIX FRAME XY'Z' AT C.O.M. "G"
  WHICH ROTATES WITH THE FRAME.

- USE POINT G AND FRAME XY'Z', CAN
  CALCULATE THE INERTIAS:
  \[ I_x = \frac{10}{3} m a^2; \quad I_{xy'} = 0; \quad I_{xz'} = 2m a^2; \ldots \]

- CAN CALCULATE THE REST, BUT THIS IS ALL
  WE NEED, SINCE

\[
\begin{bmatrix}
I_x & I_{xy'} & I_{xz'} \\
I_{xy'} & 0 & X \\
I_{xz'} & X & 0
\end{bmatrix}
\begin{bmatrix}
\omega \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
I_x w \\
0 \\
I_{xz'} w
\end{bmatrix}
\]

\[
H_G = I_G W_G =
\]

\[
\begin{bmatrix}
I_x & I_{xy'} & I_{xz'} \\
I_{xy'} & 0 & X \\
I_{xz'} & X & 0
\end{bmatrix}
\begin{bmatrix}
\omega \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
I_x w \\
0 \\
I_{xz'} w
\end{bmatrix}
\]

\[
\therefore \text{CAN EASILY SEE THAT } H_G \text{ AND } W_G
\]

\[ \text{ARE NOT ALIGNED} \Rightarrow \text{TYPICAL OF 3D ROTATIONS BUT RARELY SEEN IN PLANAR PROBLEMS} \]
• TO FIND REACTIONS, NEED TO FIND $M_G$

$M_G = \dot{H}_G = \dot{H}_{xG}^T + w_x^T H_G$

$$M_G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -w \\ 0 & w & 0 \end{bmatrix} \begin{bmatrix} I_{xw} \\ 0 \\ I_{xz'}w \end{bmatrix} = \begin{bmatrix} 0 \\ -I_{xw} \cdot w^2 \\ 0 \end{bmatrix}$$

• SO $M_G$ IS NON-ZERO DUE TO THE NON-ZERO CROSS-MOMENT $I_{xz'}$

- CAN THEN FIND THE REACTIONS AT POINTS A, B TO APPLY THIS MOMENT ABOUT THE $y'$-AXIS

$F_A = -F_B = \frac{1}{2} Mq \cdot w^2 \cdot \text{IN Z' DIRECTION}$

• FORCE COUPLE IN Z' DIRECTION, WHICH Rotates WITH THE FRAME.

• NOTE: IT IS THIS TYPE OF IMBALANCE IN A CRANKSHAFT THAT CAN CAUSE DAMAGE TO THE MOUNT.
STABILITY OF TORQUE FREE MOTION

- CAN GAIN A LOT OF INSIGHT BY CONSIDERING SPECIFIC TYPES OF MOTIONS, AND THEN SEEING HOW THE VEHICLE'S MOTION WOULD RESPOND TO SMALL PERTURBATIONS
  - MOTION "SIMILAR" TO INITIAL MOTION ⇒ STABLE

- CONSIDER ROTATION ABOUT ONE PRINCIPAL AXIS

\[ \mathbf{\bar{w}} = \mathbf{w} \hat{i} \] => \[ \mathbf{\bar{w}}_B = \begin{bmatrix} w_0 \\ 0 \\ 0 \end{bmatrix} \]

\((x, y, z) \sim BODY FRAME\).

- NOW ADD A SLIGHT PERTURBATION TO THIS MOTION
  - ASSUME TORQUE FREE

\[ \Rightarrow \mathbf{w}_B = \begin{bmatrix} w_0 + \delta w_x \\ \delta w_y \\ \delta w_z \end{bmatrix} \] PERTURBATION TO \(w_x\) WILL LEAD TO (HOPEFULLY) SMALL CHANGES TO \(w_y, w_z\)

\[ \Rightarrow \text{NEED TO FIND A WAY TO PREDICT} \]
\(\delta w_x(t), \delta w_y(t), \delta w_z(t)\)

**NOTE:** PERTURBED MOTION MUST SATISFY EULER'S EQUATIONS.
PERTURBED, TORQUE FREE EULER'S:

1) \[ \dot{\omega} = I_{xx} (\omega_0 + \delta \omega_x) + (I_{zz} - I_{yy}) \dot{\omega}_y \delta \omega_z \]

2) \[ \dot{\omega} = I_{yy} \dot{\omega}_y + (I_{xx} - I_{zz})(\omega_0 + \delta \omega_x) \delta \omega_z \]

3) \[ \dot{\omega} = I_{zz} \dot{\omega}_z + (I_{yy} - I_{xx})(\omega_0 + \delta \omega_x) \delta \omega_y \]

KEY POINTS:
- \( \omega_0 \) CONSTANT
- \( \omega_0 \gg \delta \omega_x, \delta \omega_y, \delta \omega_z \)

LINEARIZE E.O.M. BY DROPPING PRODUCTS OF \( \dot{\omega} \)'s

E.G. IN 1) \[ \dot{\omega}_y \delta \omega_z \]

IN 2) \[ (\omega_0 + \delta \omega_x) \delta \omega_z = \omega_0 \delta \omega_z + \delta \omega_x \delta \omega_z \]

LINEARIZED FORM:

\[ \delta \omega_x = 0 \quad \rightarrow \quad \delta \omega_x \text{ CONSTANT} \]

\[ I_{yy} \delta \omega_y + (I_{xx} - I_{zz}) \omega_0 \delta \omega_z = 0 \]

\[ I_{zz} \delta \omega_z + (I_{yy} - I_{xx}) \omega_0 \delta \omega_y = 0 \]

COMBINE: (DIFFERENTIATE FIRST ONE)

\[ \ddot{\delta \omega}_y + \left[ \frac{\omega_0^2 (I_{xx} - I_{yy})(I_{xx} - I_{zz})}{I_{yy} I_{zz}} \right] \delta \omega_y = 0 \]
• DIFFERENTIAL EQUATION \( \dot{\delta w} + A \dot{\delta w} = 0 \)
  \[ \Rightarrow \dot{\delta w} = B_1 e^{t\sqrt{-A}} + B_2 e^{-t\sqrt{-A}} \]

1. IF \( A > 0 \), \(-A < 0 \) \( \Rightarrow \delta w \) **SINUSOIDAL**
   \( \Rightarrow \delta w_1, \delta w_2 \) **TEND TO OSCILLATE**
   \( \sim \) **STABLE (NEUTRAL)**

2. IF \( A < 0 \), \(-A > 0 \) \( \Rightarrow \) **EXPONENTIAL GROWTH**
   IN \( \delta w \sim e^{xt} \)
   \( \therefore \) **UNSTABLE**

• IN OUR CASE

\[
A = \omega_0^2 \frac{(I_{xx} - I_{yy})(I_{xx} - I_{zz})}{I_{yy}I_{zz}}
\]

FOR \( A > 0 \), NEED:

i) \( I_{xx} > I_{yy} \) AND \( I_{xx} > I_{zz} \)

ii) \( I_{yy} > I_{xx} \) AND \( I_{zz} > I_{xx} \) **STABLE CASES**

i) CORRESPONDS TO \( I_{xx} \) BEING **LARGEST**
   **MOMENT OF INERTIA**

ii) CORRESPONDS TO \( I_{xx} \) BEING THE **SMALLEST**.

• RECALL THAT WE ARE SPINNING ABOUT X-AXIS.
• **Observations:**
  
  - If initial spin about an intermediate axis of inertia \((I_{yy} > I_{xx} > I_{zz})\) then spin unstable.
  
  - Spin about max/min axes of inertia are stable (only "neutral")

**Example**

• **Further Thoughts:**
  
  - Rotational kinetic energy \(T = \frac{1}{2} \mathbf{w}_B^T \mathbf{I}_B \mathbf{w}_B\)

  \[
  \mathbf{w}_B = \begin{bmatrix} \mathbf{w}_o \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad T_{\text{Rot}} = \frac{1}{2} I_{xx} w_o^2
  \]

  \[
  H = I_{xx} w_o
  \]

• With no external moments, \(H\) fixed

  \[
  \Rightarrow \quad T_{\text{Rot}} = \frac{1}{2} \frac{H^2}{I_{xx}} \propto \frac{1}{I_{xx}}
  \]

  \[
  \therefore \text{If } I_{xx} \text{ minimum inertia, then } T_{\text{Rot}} \text{ is the maximum value possible.}
  \]

  \[
  \Rightarrow \quad I_{xx} \sim \text{maximum } \Rightarrow \ T \sim \text{min value.}
  \]
EXAMPLE: IF \( l_x < l_z < l_y \), WHAT IS THE ORDERING OF \( I_x, I_y, I_z \) ?

![Diagram showing three blocks labeled A, B, and C with dimensions and orientations.]

\[ I_{xx} = \frac{m}{12} \left( l_y^2 + l_z^2 \right) ; \quad I_{yy} = \frac{m}{12} \left( l_x^2 + l_z^2 \right) \]

\[ I_{zz} = \frac{m}{12} \left( l_x^2 + l_y^2 \right) \]

\[ \Rightarrow I_{yy} - I_{zz} = \frac{m}{12} \left( l_z^2 - l_y^2 \right) < 0 \quad \therefore I_{yy} < I_{zz} \]

- CAN SHOW \( I_{yy} < I_{zz} < I_{xx} \) \( \{ \) CONSISTENT WITH VISUAL "INSPECTION" OF MASS DISTRIBUTION \( \}

- SPIN STABILITY ?

- MUCH MORE ON THIS TYPE OF PROBLEM LATER ON.
- So, if there is an energy dissipation mechanism in the system, expect $T_{rot}$ to reduce $\Rightarrow$ only rotations about the maximum axis are stable
- Spin about min axis degenerates

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**INTERNAL

- CIRCULAR CYLINDER
  \[ I_x \sim \frac{1}{2} ma^2 \]
  \[ I_y, I_z = \frac{m}{12} (3a^2 + L^2) \]

\[ \therefore \text{for long cylinder, } I_x < I_y = I_z \]
**INFAMOUS EXAMPLE: EXPLORER 1**

- **PLAN WAS TO SPIN SATELLITE ABOUT LONG AXIS** → **STABILITY (?) → MIN AXIS**.

- **BUT THE ANTENNAS DISSIPATED ENERGY**
  → **MIN AXIS SPIN UNSTABLE**
  → **BODY STARTED TO TUMBLE**
  → **STABILIZED IN SPIN ABOUT MAJOR AXIS**

- **FOR STABLE SPIN, FLY A PLATE:**

   ![Diagram of a satellite spinning horizontally with a plate attached to its side. The plate is labeled with an arrow suggesting it is spinning along the satellite's axis.]
FURTHER INSIGHTS ON TORQUE
FREE MOTION - GEOMETRIC

\[ M = 0 \]

- TORQUE FREE - \( \dot{H} \) CONSTANT
  - \( |H_B| \) IS CONSTANT, BUT \( H_B \) CAN CHANGE.

- CAN ALSO SHOW THAT ROTATIONAL KINETIC ENERGY IS ALSO CONSTANT, I.E. \( \dot{T}_{\text{rot}} = 0 \)

  WHY? \[ T_{\text{rot}} = \frac{1}{2} \dot{\omega} \cdot \dot{\omega} \]

  \[ \Rightarrow \dot{T}_{\text{rot}} = \dot{\omega} \cdot \dot{\omega} = 0 \]

  BUT RECALL THAT \( \dot{\omega} \equiv \dot{\omega}_I; \ddot{\omega} \equiv \ddot{\omega}_I \)

  \[ \dot{\omega}_I = \dot{\omega} \times \dot{\omega}_I \]

  \[ \Rightarrow \dot{T}_{\text{rot}} = \dot{\omega} \cdot \dot{\omega}_I = -\dot{\omega} \times (\dot{\omega}_I \cdot \dot{\omega}) \]

  \[ \Rightarrow \dot{T}_{\text{rot}} = \dot{\omega} \cdot (-\omega \times (\dot{\omega}_I \cdot \dot{\omega})) + \dot{\omega} \cdot (\dot{\omega}_I \times \dot{\omega}) = 0 \]

- RECALL THAT \( \dot{\omega} \times \dot{\omega} \) PERPENDICULAR TO BOTH \( \dot{\omega} \) AND \( \dot{\omega}_I \), SO \( \dot{\omega} \cdot (\dot{\omega}_I \times \dot{\omega}) = 0 \)

  \[ \therefore \dot{T}_{\text{rot}} = 0 \Rightarrow T_{\text{rot}} \text{ CONSTANT.} \]

- ASSUMES THAT THERE ARE NO INTERNAL DISSIPATION MECHANISMS, AS DISCUSSED BEFORE.
* Now assume that the body XYZ axes are aligned with the principal axes

\[ T_{\text{rot}} = \frac{1}{2} \mathbf{\omega} \cdot \mathbf{H} = \frac{1}{2} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \]

\[ \Rightarrow 2 T_{\text{rot}} = I_{xx} w_x^2 + I_{yy} w_y^2 + I_{zz} w_z^2 \]

* This is equivalent to a constraint equation on the allowable combinations of \( w_x, w_y, w_z \) for a given \( T_{\text{rot}} \)

- Set of possible values of \( w_x, w_y, w_z \) are on an ellipsoid aligned with the principal axes.

* More on 10-20

* Ellipsoid size in each direction \( \sim \sqrt{\frac{2 T_{\text{rot}}}{I_{xx}}} \)

So large \( I_{xx} \) \( \rightarrow \) ellipsoid small in that direction.
• **MOMENTUM:** \( \vec{H} \) FIXED, So \( |\vec{H}|^2 \) MUST BE CONSTANT

\[
H_8 = \begin{bmatrix} I_{xx} & 0 \\ I_{yy} & 0 \\ 0 & I_{zz} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} I_{xx} w_x \\ I_{yy} w_y \\ I_{zz} w_z \end{bmatrix}
\]

\[|H_8|^2 = I_{xx} w_x^2 + I_{yy} w_y^2 + I_{zz} w_z^2 = C\]

- PROVIDES ANOTHER CONSTRAINT ON THE POSSIBLE COMBINATIONS OF \( w_x, w_y, w_z \)

• **MOTION OF THE BODY MUST SATISFY BOTH CONSTRAINTS**

  ➔ FEASIBLE ANGULAR VELOCITY COMBINATIONS \( w_x, w_y, w_z \) AS SEEN IN THE BODY FRAME

  ➔ MUST LIE AT THE INTERSECTION OF THE TWO ELLIPSOIDS.

  ➔ CALLED A **POLHODE**

  ➔ INTERSECTION CHANGES DEPENDING ON \( T_{\text{rot}} \) AND \( |H_8| \).
\[
\text{ENERGY} \quad \frac{w_x^2}{2T/I_{xx}} + \frac{w_y^2}{2T/I_{yy}} + \frac{w_z^2}{2T/I_{zz}} = 1
\]

\[
\text{MOMENTUM} \quad \frac{w_x^2}{|Hb|^2/I_{xx}} + \frac{w_y^2}{|Hb|^2/I_{yy}} + \frac{w_z^2}{|Hb|^2/I_{zz}} = 1
\]

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**Figure adapted from P.C. Hughes, *Spacecraft Attitude Dynamics* (John Wiley and Sons, 1986)**
High Energy Case: $I_{xx} = 3$ $I_{yy} = 4$ $I_{zz} = 7$
Med High Energy Case: $I_{xx} = 3$  $I_{yy} = 4$  $I_{zz} = 7$
Med Low Energy Case: \( I_{xx} = 3 \quad I_{yy} = 4 \quad I_{zz} = 7 \)
Low Energy Case: $I_{xx} = 3 \quad I_{yy} = 4 \quad I_{zz} = 7$