Lecture AC 2

Aircraft Longitudinal Dynamics

- Typical aircraft open-loop motions
- Longitudinal modes
- Impact of actuators
- **Linear Algebra in Action!**
Longitudinal Dynamics

- For notational simplicity, let $X = F_x$, $Y = F_y$, and $Z = F_z$

  $X_u \equiv \left( \frac{\partial F_x}{\partial u} \right), \ldots$

- Longitudinal equations (1–15) can be rewritten as:

  $m \dot{u} = X_u u + X_w w - mg \cos \Theta_0 \theta + \Delta X_c$

  $m(\dot{w} - q U_0) = Z_u u + Z_w w + Z_{\dot{w}} \dot{w} + Z_{q} q - mg \sin \Theta_0 \theta + \Delta Z_c$

  $I_{yy} \dot{q} = M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q + \Delta M_c$

  - There is no roll/yaw motion, so $q = \dot{\theta}$.
  - The control commands $\Delta X_c \equiv \Delta F_c^x$, $\Delta Z_c \equiv \Delta F_c^z$, and $\Delta M_c \equiv \Delta M^c$
    have not yet been specified.

- Rewrite in state space form as

\[
\begin{bmatrix}
  m \dot{u} \\
  (m - Z_{\dot{w}}) \dot{w} \\
  -M_{\dot{w}} \dot{w} + I_{yy} \dot{q} \\
  \dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
  X_u & X_w & 0 & -mg \cos \Theta_0 \\
  Z_u & Z_w & Z_q + m U_0 & -mg \sin \Theta_0 \\
  M_u & M_w & M_q & 0 \\
  0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  u \\
  w \\
  q \\
  \theta
\end{bmatrix}
+ \begin{bmatrix}
  \Delta X_c \\
  \Delta Z_c \\
  \Delta M_c
\end{bmatrix}
\]

\[
\begin{bmatrix}
  m & 0 & 0 & 0 \\
  0 & m - Z_{\dot{w}} & 0 & 0 \\
  0 & -M_{\dot{w}} & I_{yy} & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \dot{u} \\
  \dot{w} \\
  \dot{q} \\
  \dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
  X_u & X_w & 0 & -mg \cos \Theta_0 \\
  Z_u & Z_w & Z_q + m U_0 & -mg \sin \Theta_0 \\
  M_u & M_w & M_q & 0 \\
  0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  u \\
  w \\
  q \\
  \theta
\end{bmatrix}
+ \begin{bmatrix}
  \Delta X_c \\
  \Delta Z_c \\
  \Delta M_c
\end{bmatrix}
\]

\[E \dot{X} = \hat{A}X + \hat{c}\quad \text{descriptor state space form}\]

\[\dot{X} = E^{-1}(\hat{A}X + \hat{c}) = AX + c\]
• Write out in state space form:

\[
A = \begin{bmatrix}
\frac{X_u}{m} & \frac{X_w}{m} & 0 & \frac{-g \cos \Theta_0}{m} \\
\frac{Z_u}{m-Z_w} & \frac{Z_w}{m-Z_w} & \frac{Z_q + mU_0}{m-Z_w} & \frac{-mq \sin \Theta_0}{m-Z_w} \\
\frac{I_{yy}^{-1} [M_u + Z_u \Gamma]}{m-Z_w} & \frac{I_{yy}^{-1} [M_w + Z_w \Gamma]}{m-Z_w} & \frac{I_{yy}^{-1} [M_q + (Z_q + mU_0) \Gamma]}{m-Z_w} & \frac{-I_{yy}^{-1} mg \sin \Theta \Gamma}{m-Z_w} \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\Gamma = \frac{M_w}{m-Z_w}
\]

• To figure out the \( c \) vector, we have to say a little more about how the control inputs are applied to the system.
Longitudinal Actuators

- Primary actuators in longitudinal direction are the elevators and the thrust.
  - Clearly the thrusters/elevators play a key role in defining the steady-state/equilibrium flight condition
  - Now interested in determining how they also influence the aircraft motion about this equilibrium condition

\[
deflect \text{ elevator} \rightarrow u(t), w(t), q(t), \ldots\]

- Recall that we defined \( \Delta X_c \) as the perturbation in the total force in the \( X \) direction as a result of the actuator commands
  - Force change due to an actuator deflection from trim

- Expand these aerodynamic terms using the same perturbation approach

\[
\Delta X_c = X_{\delta_c} \delta_c + X_{\delta_p} \delta_p
\]

- \( \delta_c \) is the deflection of the elevator from trim (down positive)
- \( \delta_p \) change in thrust
- \( X_{\delta_c} \) and \( X_{\delta_p} \) are the control stability derivatives
Now we have that

\[
\mathbf{c} = E^{-1} \begin{bmatrix}
\Delta X_c \\
\Delta Z_c \\
\Delta M_c \\
0
\end{bmatrix} = E^{-1} \begin{bmatrix}
X_{\delta_e} & X_{\delta_p} \\
Z_{\delta_e} & Z_{\delta_p} \\
M_{\delta_e} & M_{\delta_p} \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\delta_c \\
\delta_p
\end{bmatrix} = Bu
\]

For the longitudinal case

\[
B = \begin{bmatrix}
\frac{X_{\delta_e}}{m} & \frac{X_{\delta_p}}{m} \\
\frac{Z_{\delta_e}}{m-Z_{\dot{w}}} & \frac{Z_{\delta_p}}{m-Z_{\dot{w}}} \\
I_{yy}^{-1} [M_{\delta_e} + Z_{\delta_e} \Gamma] & I_{yy}^{-1} [M_{\delta_p} + Z_{\delta_p} \Gamma] \\
0 & 0
\end{bmatrix}
\]

Typical values for the B747

\[
\begin{align*}
X_{\delta_e} &= -16.54 \\
X_{\delta_p} &= 0.3mg = 849528 \\
Z_{\delta_e} &= -1.58 \cdot 10^6 \\
Z_{\delta_p} &\approx 0 \\
M_{\delta_e} &= -5.2 \cdot 10^7 \\
M_{\delta_p} &\approx 0
\end{align*}
\]

Aircraft response \( y = G(s)u \)

\[
\dot{X} = AX + Bu \rightarrow G(s) = C(sI - A)^{-1}B \\
y = CX
\]

We now have the means to modify the dynamics of the system, but first let’s figure out what \( \delta_e \) and \( \delta_p \) really do.
Longitudinal Response

- **Final response** to a step input \( u = \hat{u}/s, \; y = G(s)u \), use the FVT

\[
\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \left( G(s) \frac{\hat{u}}{s} \right)
\]

\[
\Rightarrow \lim_{t \to \infty} y(t) = G(0)\hat{u} = -(CA^{-1}B)\hat{u}
\]

- **Initial response** to a step input, use the IVT

\[
y(0^+) = \lim_{s \to \infty} s \left( G(s) \frac{\hat{u}}{s} \right) = \lim_{s \to \infty} G(s)\hat{u}
\]

- For your system, \( G(s) = C(sI - A)^{-1}B + D \), but \( D \equiv 0 \), so

\[
\lim_{s \to \infty} G(s) \to 0
\]

- **Note:** there is NO immediate change in the output of the motion variables in response to an elevator input \( \Rightarrow y(0^+) = 0 \)

- Consider the *rate of change* of these variables \( \dot{y}(0^+) \)

\[
\dot{y}(t) = C\dot{X} = CAX + CBu
\]

and normally have that \( CB \neq 0 \). Repeat process above\(^1\) to show that \( \dot{y}(0^+) = CB\hat{u} \), and since \( C \equiv I \),

\[
\dot{y}(0^+) = B\hat{u}
\]

- Looks good. Now compare with numerical values computed in MATLAB®

Plot \( u, \alpha \), and flight path angle \( \gamma = \theta - \alpha \) (since \( \Theta_0 = \gamma_0 = 0 \))

See AC 1-10

\(^1\)Note that \( C(sI - A)^{-1}B + D = D + \frac{CB}{s} + \frac{CA^{-1}B}{s^2} + \ldots \)
Elevator (1° elevator down – stick forward)

- See very rapid response that decays quickly (mostly in the first 10 seconds of the $\alpha$ response)
- Also see a very lightly damped long period response (mostly $u$, some $\gamma$, and very little $\alpha$). Settles in >600 secs
- Predicted steady state values from code:

  14.1429 m/s  $u$  (speeds up)
  -0.0185 rad  $\alpha$  (slight reduction in AOA)
  -0.0000 rad/s  $q$
  -0.0161 rad  $\theta$
  0.0024 rad  $\gamma$

  - Predictions appear to agree well with the numerical results.
  - Primary result is a slightly lower angle of attack and a higher speed

- Predicted initial rates of the output values from code:

  -0.0001 m/s$^2$  $\dot{u}$
  -0.0233 rad/s  $\dot{\alpha}$
  -1.1569 rad/s$^2$  $\dot{q}$
  0.0000 rad/s  $\dot{\theta}$
  0.0233 rad/s  $\dot{\gamma}$

  - All outputs are at zero at $t = 0^+$, but see rapid changes in $\alpha$ and $q$.
  - Changes in $u$ and $\gamma$ (also a function of $\theta$) are much more gradual – not as easy to see this aspect of the prediction

- Initial impact  Change in $\alpha$ and $q$ (pitches aircraft)
- Long term impact  Change in $u$ (determines speed at new equilibrium condition)
Thrust (1/6 input)

- Motion now dominated by the lightly damped long period response
- Short period motion barely noticeable at beginning.
- Predicted steady state values from code:

\[
\begin{align*}
0 & \text{ m/s } u \\
0 & \text{ rad } \alpha \\
0 & \text{ rad/s } q \\
0.05 & \text{ rad } \theta \\
0.05 & \text{ rad } \gamma
\end{align*}
\]

- Predictions appear to agree well with the simulations.
- Primary result is that we are now climbing with a flight path angle of 0.05 rad at the same speed we were going before.

- Predicted initial rates of the output values from code:

\[
\begin{align*}
2.9430 & \text{ m/s}^2 \dot{u} \\
0 & \text{ rad/s } \dot{\alpha} \\
0 & \text{ rad/s}^2 \dot{q} \\
0 & \text{ rad/s } \dot{\theta} \\
0 & \text{ rad/s } \dot{\gamma}
\end{align*}
\]

- Changes to \( \alpha \) are very small, and \( \gamma \) response initially flat.
- Increase power, and the aircraft initially speeds up

- **Initial impact** Change in \( u \) (accelerates aircraft)
- **Long term impact** Change in \( \gamma \) (determines climb rate)
Figure 1: Step Response to 1 deg elevator perturbation – B747 at M=0.8
Figure 2: Step Response to 1/6 thrust perturbation – B747 at M=0.8
• Summary:

- To increase equilibrium climb rate, add power.

- To increase equilibrium speed, increase $\delta_e$ (move elevator further down).

- Transient (initial) effects are the opposite and tend to be more consistent with what you would intuitively expect to occur.
Modal Behavior

- Analyze the model of the vehicle dynamics to quantify the responses we saw.
  - Homogeneous dynamics are of the form \( \dot{X} = AX \), so the response is
    \[ X(t) = e^{At}X(0) \] – a matrix exponential.

- To simplify the investigation of the system response, find the **modes** of the system using the **eigenvalues** and **eigenvectors**
  - \( \lambda \) is an **eigenvalue** of \( A \) if
    \[ \det(\lambda I - A) = 0 \]
    which is true iff there exists a nonzero \( v \) (**eigenvector**) for which
    \[ (\lambda I - A)v = 0 \Rightarrow Av = \lambda v \]
  - If \( A \) \((n \times n)\), typically will get \( n \) eigenvalues and eigenvectors \( Av_i = \lambda_i v_i \)
  - Assuming that the eigenvectors are **linearly independent**, can form
    \[ A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & & \lambda_n \end{bmatrix} \]
    \[ AT = T\Lambda \]
    \[ \Rightarrow T^{-1}AT = \Lambda \quad \text{,} \quad A = T\Lambda T^{-1} \]
  - Given that \( e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots \), and that \( A = T\Lambda T^{-1} \), then it is easy to show that
    \[ X(t) = e^{At}X(0) = Te^{\Lambda t}T^{-1}X(0) = \sum_{i=1}^{n} v_i e^{\lambda_i t} \beta_i \]
  - **State solution** is a **linear combination** of the **system modes** \( v_ie^{\lambda_i t} \)

\( e^{\lambda_i t} \) – determines the **nature** of the time response
\( v_i \) – determines the extent to which each state **contributes** to that mode
\( \beta_i \) – determines the extent to which the initial condition **excites** the mode
• Thus the total behavior of the system can be found from the system modes

• Consider numerical example of B747

\[
A = \begin{bmatrix}
-0.0069 & 0.0139 & 0 & -9.8100 \\
-0.0905 & -0.3149 & 235.8928 & 0 \\
0.0004 & -0.0034 & -0.4282 & 0 \\
0 & 0 & 1.0000 & 0
\end{bmatrix}
\]

which gives two sets of complex eigenvalues

\[
\lambda = -0.3717 \pm 0.8869i, \quad \omega = 0.962, \quad \zeta = 0.387, \quad \text{short period}
\]

\[
\lambda = -0.0033 \pm 0.0672i, \quad \omega = 0.067, \quad \zeta = 0.049, \quad \text{Phugoid - long period}
\]

- result is consistent with step response - heavily damped fast response, and a lightly damped slow one.

• To understand the eigenvectors, we have to do some normalization (scales each element appropriately so that we can compare relative sizes)

- \( \hat{u} = u/U_0, \hat{w} = w/U_0, \hat{q} = q/(2U_0/c) \)

- Then divide through so that \( \theta \equiv 1 \)

<table>
<thead>
<tr>
<th>Short Period</th>
<th>Phugoid</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{u} )</td>
<td>0.0156 + 0.0244i</td>
</tr>
<tr>
<td>( \hat{w} )</td>
<td>1.0202 + 0.3553i</td>
</tr>
<tr>
<td>( \hat{q} )</td>
<td>-0.0066 + 0.0156i</td>
</tr>
<tr>
<td>( \theta )</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

• **Short Period** – primarily \( \theta \) and \( \alpha = \hat{w} \) in the same phase. The \( \hat{u} \) and \( \hat{q} \) response is very small.

• **Phugoid** – primarily \( \theta \) and \( \hat{u} \), and \( \theta \) lags by about 90°. The \( \hat{w} \) and \( \hat{q} \) response is very small \( \Rightarrow \) consistent with approximate solution on AC 2–1?

• Dominant behavior agrees with time step responses – note how initial conditions were formed.
Figure 3: Mode Response – B747 at M=0.8
Summary

- Two primary longitudinal modes: phugoid and short-period

- Impact of the various actuators clarified:
  - Short time-scale
  - Long time-scale

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