Problem 1 - Solitons (50p)

Problem Statement

J. Scott Russell wrote in 1844:

“I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.”

In 1895, Korteweg and de Vries formulated the equation

\[ u_t - 6uu_x + u_{xxx} = 0, \]  

(1)

which models Russell’s observation. The term \( uu_x \) describes the sharpening of the wave and \( u_{xxx} \) the dispersion (i.e., waves with different wave lengths propagate with different velocities). The balance between these two terms allows for a propagating wave with unchanged form. The primary application of solitons today are in optical fibers, where the linear dispersion of the fiber provides smoothing of the wave, and the non-linear properties give the sharpening. The result is a very stable and long-lasting pulse that is free from dispersion, which is a problem with traditional optical communication techniques.

Questions

1) (5p) Show using direct substitution that the one-soliton solution

\[ u_1(x, t) = -\frac{v}{2\cosh^2\left(\frac{1}{2\sqrt{v}}(x - vt - x_0)\right)} \]  

(2)

solves the KdV equation (1). Here, \( v > 0 \) and \( x_0 \) are arbitrary parameters.
2) (10p) We will solve the KdV equation numerically using the method of lines and finite difference approximations for the space derivatives. Rewrite the equation as

\[ \frac{\partial u}{\partial t} = 6uu_x - u_{xxx}, \] (3)

and derive a second-order accurate difference approximation of the right-hand side.

3) (15p) For the time integration, we will use a fourth order Runge-Kutta scheme:

\[ \alpha^1 = \Delta t f(u^i) \] (4)
\[ \alpha^2 = \Delta t f(u^i + \alpha^1/2) \] (5)
\[ \alpha^3 = \Delta t f(u^i + \alpha^2/2) \] (6)
\[ \alpha^4 = \Delta t f(u^i + \alpha^3) \] (7)
\[ u^{i+1} = u^i + \frac{1}{6}(\alpha^1 + 2\alpha^2 + 2\alpha^3 + \alpha^4). \] (8)

The stability region for this scheme consists of all \( z \) such that

\[ |1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}| \leq 1. \] In particular, all points on the imaginary axis between \( \pm i2\sqrt{2} \) are included.

Our equation (1) is non-linear, and to make a stability analysis we first have to linearize it. In this case, it turns out that the stability will be determined by the discretization of the third-derivative term \( u_{xxx} \). Therefore, consider the simplified problem

\[ \frac{\partial u}{\partial t} = -u_{xxx}, \] (9)

and use von Neumann stability analysis to derive an expression for the maximum allowable time-step \( \Delta t \) in terms of \( \Delta x \).

4) (20p) Write a program that solves the equation using your discretization. Solve it in the region \(-8 \leq x \leq 8\) with a grid size \( \Delta x = 0.1 \), and use periodic boundary conditions:

\[ x(-8) = x(8). \] (10)

Integrate from \( t = 0 \) to \( t = 2 \), using an appropriate time-step that satisfies the condition you derived above. For each of the initial conditions below, plot the solution at \( t = 2 \) and comment on the results.

a. To begin with, use a single soliton (2) as initial condition, that is, \( u(x, 0) = u_1(x, 0) \). Set \( v = 16 \) and \( x_0 = 0 \).

b. The one-soliton solution looks almost like a Gaussian. Try \( u(x, 0) = -8e^{-x^2} \).

c. Try the two-soliton solution \( u(x, 0) = -6/\cosh^2(x) \).

d. Create “your own” two-soliton solution by superposing two one-soliton solutions with \( v = 16 \) and \( v = 4 \) (both with \( x_0 = 0 \)).

e. Same as before, but with \( v = 16, x_0 = 4 \) and \( v = 4, x_0 = -4 \). Describe what happens when the two solitons cross (amplitudes, velocities), and after they have crossed.
Problem 2 - Traffic Flow (50p)

Problem Statement

Consider the traffic flow problem, described by the non-linear hyperbolic equation:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0
\] (11)

with \( \rho = \rho(x, t) \) the density of cars (vehicles/km), and \( u = u(x, t) \) the velocity. Assume that the velocity \( u \) is given as a function of \( \rho \):

\[
u = u_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right).
\] (12)

With \( u_{\text{max}} \) the maximum speed and \( 0 \leq \rho \leq \rho_{\text{max}} \). The flux of cars is therefore given by:

\[
\phi(\rho) = \rho u_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right).
\] (13)

We will solve this problem using a first order finite volume scheme:

\[
\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \left( \phi_i^{n+1} - \phi_i^n \right).
\] (14)

For the numerical flux function, we will consider two different schemes:

a) Roe’s Scheme

The expression of the numerical flux is given by:

\[
\phi_{\text{R}}^{i+\frac{1}{2}} = \frac{1}{2} \left[ \phi(i) + \phi(i+1) \right] - \frac{1}{2} \left| a_{i+\frac{1}{2}} \right| (\rho_{i+1} - \rho_i)
\] (15)

with

\[
a_{i+\frac{1}{2}} = u_{\text{max}} \left( 1 - \frac{\rho_i + \rho_{i+1}}{\rho_{\text{max}}} \right).
\] (16)

Note that \( a_{i+\frac{1}{2}} \) satisfies

\[
f(\rho_{i+1}) - f(\rho_i) = a_{i+\frac{1}{2}} (\rho_{i+1} - \rho_i).
\] (17)

b) Godunov’s Scheme

In this case the numerical flux is given by:

\[
\phi_{\text{G}}^{i+\frac{1}{2}} = f \left( \rho \left( x_{i+\frac{1}{2}}, t^{n+1} \right) \right) = \begin{cases} 
\min_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\
\max_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1}.
\end{cases}
\] (18)
Questions

1) (25p) For both Roe’s Scheme and Godunov’s Scheme, look at the problem of a traffic light turning green at time \( t = 0 \). We are interested in the solution at \( t = 2 \) using both schemes. What do you observe for each of the schemes? Explain briefly why the behavior you get arises.

Use the following problem parameters:

\[
\rho_{\text{max}} = 1.0, \quad \rho_L = 0.8 \\
u_{\text{max}} = 1.0 \\
\Delta x = \frac{4}{400}, \quad \Delta t = \frac{0.8 \Delta x}{u_{\text{max}}} \\
\tag{19}
\]

The initial condition at the instant when the traffic light turns green is

\[
\rho(0) = \begin{cases} 
\rho_L, & x < 0 \\
0, & x \geq 0 
\end{cases} \\
\tag{20}
\]

For the rest of this problem use only the scheme(s) which are valid models of the problem.

2) (25p) Simulate the effect of a traffic light at \( x = -\frac{\Delta x}{2} \) which has a period of \( T = T_1 + T_2 = 2 \) units. Assume that the traffic light is \( T_1 = 1 \) units on red and \( T_2 = 1 \) units on green. Assume a sufficiently high flow density of cars (e.g. set \( \rho = \frac{\rho_{\text{max}}}{2} \) on the left boundary – giving a maximum flux), and determine the average flow, or capacity of cars over a time period \( T \).

The average flow can be approximated as

\[
\dot{q} = \frac{1}{N_T} \sum_{n=1}^{N_T} f^n = \frac{1}{N_T} \sum_{n=1}^{N_T} \rho^n u^n, \\
\tag{21}
\]

where \( N_T \) is the number of time steps for each period \( T \). You should run your computation until \( \dot{q} \) over a time period does not change. Note that by continuity \( \dot{q} \) can be evaluated over any point in the interior of the domain (in order to avoid boundary condition effects, we consider only those points on the interior domain).

Note: A red traffic light can be modeled by simply setting \( F_{i+\frac{1}{2}} = 0 \) at the position where the traffic light is located.

3) BONUS: (Possible +10p\(^1\)) Assume now that we simulate two traffic lights, one located at \( x = 0 \), and the other at \( x = 0.15 \), both with a period \( T \). Calculate the road capacity (= average flow) for different delay factors. That is if the first light turns green at time \( t \), then the second light will turn green at \( t + \tau \). Solve for \( \tau = k \frac{T}{10}, \quad k = 0, \ldots, 9 \). Plot your results of capacity vs \( \tau \) and determine the optimal delay \( \tau \).

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\(^1\)Only applied to gain a maximum of 100%. Additional bonus points are not carried over to future assignments.