# New Schedule

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Fall 2004 Pattern Recognition for Vision
9.913 Pattern Recognition for Vision

Classification
Bernd Heisele
Overview

Introduction
Linear Discriminant Analysis
Support Vector Machines
Literature & Homework
Introduction

Classification

Two approaches:
- Linear, non-linear separation
- Two class, multi-class problems

Two approaches:
- Density estimation, classify with Bayes decision:
  Linear Discr. Analysis (LDA), Quadratic Discr. Analysis (QDA)
- Without density estimation: Support Vector Machines (SVM)
LDA Bayes Rule

Bayes Rule

\[ p(x, \omega) = p(x | \omega)P(\omega) = P(\omega | x)p(x) \quad \Rightarrow \]

\[ P(\omega | x) = \frac{p(x | \omega)P(\omega)}{p(x)} \]

\( x \): random vector

\( \omega \): class
LDA— Bayes Decision Rule

Decide $\omega_1$ if \[ \frac{P(\omega_1 \mid x)}{P(\omega_2 \mid x)} > 1 ; \text{ otherwise } \omega_2 \]

Likelihood Ratio
\[ \frac{p(x \mid \omega_1)P(\omega_1)}{p(x \mid \omega_2)P(\omega_2)} > 1 \]

Log Likelihood Ratio
\[ \ln \left( \frac{p(x \mid \omega_1)P(\omega_1)}{p(x \mid \omega_2)P(\omega_2)} \right) > 0, \ln \left( \frac{p(x \mid \omega_1)}{p(x \mid \omega_2)} \right) + \ln \left( \frac{P(\omega_1)}{P(\omega_2)} \right) > 0 \]
LDA—Two Classes, Identical Covariance

Gaussian: \( p(x \mid \omega_i) = \frac{1}{(2\pi)^{d/2} |\Sigma_i|^{1/2}} e^{-\frac{1}{2}(x-\mu_i)^T\Sigma_i^{-1}(x-\mu_i)} \)

assume identical covariance matrices \( \Sigma_1 = \Sigma_2 \):

\[
\ln \left( \frac{p(x \mid \omega_1)}{p(x \mid \omega_2)} \right) + \ln \left( \frac{P(\omega_1)}{P(\omega_2)} \right)
= \frac{1}{2} (x - \mu_2)^T S^{-1} (x - \mu_2) - \frac{1}{2} (x - \mu_1)^T S^{-1} (x - \mu_1) + \ln \left( \frac{P(\omega_1)}{P(\omega_2)} \right)
= x^T S^{-1} (\mu_1 - \mu_2) + \frac{1}{2} (\mu_1 + \mu_2)^T S^{-1} (\mu_2 - \mu_1) + \ln \left( \frac{P(\omega_1)}{P(\omega_2)} \right)
= x^T w + b \text{ linear decision function: } \omega_1 \text{ if } x^T w + b > 0
LDA—Two Classes, Identical Covariance

\[ f(x) = x^T S^{-1}(\mu_1 - \mu_2) + \frac{1}{2}(\mu_1 + \mu_2)^T S^{-1}(\mu_2 - \mu_1) + \ln(P(\omega_1) / P(\omega_2)) \]

\[ = x'^T (\mu'_1 - \mu'_2) + \frac{1}{2}(\mu'_1 + \mu'_2)^T (\mu'_2 - \mu'_1) + \ln(P(\omega_1) / P(\omega_2)) \]

\[ \Gamma^T \Gamma = S^{-1}, \quad x'^T = x^T \Gamma^T, \quad x' = \Gamma x, \quad \mu'_1 - \mu'_2 = \Gamma (\mu_1 - \mu_2) \]

\[ S = UDU^T, \quad S^{-1} = UD^{-1}U^T = \Gamma^T \Gamma, \quad \Gamma = D^{-1/2}U^T \]
LDA—Computation

$$\hat{\mu}_i = \frac{1}{N_i} \sum_{n=1}^{N_i} x_{i,n}$$

$$\hat{S} = \frac{1}{N_1 + N_2} \sum_{i=1}^{2} \sum_{n=1}^{N_i} (x_{i,n} - \hat{\mu}_i)(x_{i,n} - \hat{\mu}_i)^T$$

Density estimation

$$f(x) = \text{sign}(x^T w + b)$$

$$w = \hat{S}^{-1}(\hat{\mu}_1 - \hat{\mu}_2)$$

$$b = \frac{1}{2} (\hat{\mu}_1 + \hat{\mu}_2)^T S^{-1} (\hat{\mu}_2 - \hat{\mu}_1) + \ln \left( \frac{P(\omega_1)}{P(\omega_2)} \right)$$

Approximate by $\frac{N_1}{N_2}$
QDA—Two classes, different covariance matrix

Quadratic Discriminant Analysis

decide $\omega_1$ if $f(x) > 0$

$$f(x) = \ln \left( p(x | \omega_1) \right) + \ln(P(\omega_1)) - \ln \left( p(x | \omega_2) \right) - \ln \left( P(\omega_2) \right)$$

$$\ln \left( p(x | \omega_1) \right) = -\frac{1}{2} \ln |\Sigma_1| - \frac{1}{2} (x - \mu_1)^T \Sigma^{-1}_1 (x - \mu_1) + \ln P(\omega_1)$$

$$f(x) = x^T A x + w^T x + w_0 \quad - \text{quadratic}$$

where

$$A = -\frac{1}{2} \left( \Sigma^{-1}_1 - \Sigma^{-1}_2 \right) \quad - \text{a matrix}$$

$$w = \Sigma^{-1}_1 \mu_1 - \Sigma^{-1}_2 \mu_2 \quad - \text{a vector}$$

$$w_o = \ldots \quad \text{well, the rest of it} \quad - \text{a scalar}$$
LDA Multiclass, Identical Covariance

Find the linear decision boundaries for $k$ classes:
For two classes we have:

$$f(x) = x^T w + b \quad \text{decide } \Omega_1 \text{ if } f(x) > 0$$

In the multi-class case we have $k - 1$ decision functions:

$$f_{1,2}(x) = x^T w_{1,2} + b_{1,2},$$

$$f_{1,3}(x) = x^T w_{1,3} + b_{1,3},$$

$$\vdots$$

$$f_{1,k}(x) = x^T w_{1,k} + b_{1,k}$$

$\Rightarrow$ we have to determine $(k - 1)(p + 1)$ parameters,

$p$ is dimension of $x$
LDA Multiclass, Identical Covariance

Find the $n$–dimensional subspace that gives the best linear discrimination between the $k$ classes.

$$y = (w_1 \mid w_2 \mid \ldots \mid w_n)^T x$$

also known as Fisher Linear Discriminant
LDA—Multiclass, Identical Covariance

Computation

compute the $d \times k$ matrix $\mathbf{M} = (\mu_1 | \mu_2 | ... | \mu_k)$ and cov. matrix $\mathbf{S}$

compute the $\mu'$: $\mathbf{M}' = \Gamma \mathbf{M}$, $\Sigma^{-1} = \Gamma^T \Gamma$

compute the cov. matrix $\mathbf{B}'$ of $\mu'$

compute the eigenvectors $\mathbf{v}_i'$ of $\mathbf{B}'$ ranked by eigenvalues

calculate $\mathbf{y}$ by projecting $\mathbf{x}$ into $X'$ and then onto the eigenvector:

$y_i = \mathbf{v}_i'^T \Gamma \mathbf{x} \Rightarrow \mathbf{w}_i = \Gamma^T \mathbf{v}_i'$
Find \( w \) such that the ratio of between-class and in-class variance is maximized if the data is projected onto \( w \):

\[
y = w^T x
\]

\[
\max \frac{w^T B w}{w^T S w}, \quad B = MM^T \quad \text{the covariance of the } \mu \text{'s}
\]

can be written as:

\[
\max w^T B w \quad \text{subject to } w^T S w = 1
\]

generalized eigenvalue problem, solution are the ranked eigenvectors of \( S^{-1} B \)

...same is an previous derivation.
Image removed due to copyright considerations. See: R. Gutierrez-Osuna
http://research.cs.tamu.edu/prism/lectures/pr/pr_l10.pdf
LDA/QDA—Summary

Advantages:
• LDA is the Bayes classifier for multivariate Gaussian distributions with common covariance.
• LDA creates linear boundaries which are simple to compute.
• LDA can be used for representing multi-class data in low dimensions.

• QDA is the Bayes classifier for multivariate Gaussian distributions.
• QDA creates quadratic boundaries.

Problems:
• LDA is based on a single prototype per class (class center) which is often insufficient in practice.
Variants of LDA

Nonparameteric LDA (Fukunaga) removes the unimodal assumption by the scatter matrix using local information. More than $k-1$ features can be extracted.

Orthonormal LDA (Okada&Tomita) computes projections that maximize separability and are pair-wise orthonormal.

Generalized LDA (Lowe) Incorporates a cost function similar to Bayes Risk minimization.

….and many many more (see “Elements of Statistical Learning” Hastie, Tibshirani, Friedman)
Find the separating function $f(x) = \text{sign}(x_i^T w + b)$ which maximizes the margin $M = 2d$ on the training data. \Rightarrow \text{Maximum margin classifier.}
SVM—Primal, (LS)

Training data consists of $N$ pairs $\{(x_1, y_1), \ldots, (x_N, y_N)\}, y_i \in \{-1, 1\}$. The problem of maximizing the margin $2d$ can be formulated as:

$$\max_{w, b} 2d \quad \text{subject to} \quad \begin{cases} y_i (x_i^T w' + b') \geq d \\ \|w'^2\| = 1 \end{cases}$$

or alternatively: $w = w' / d, b = b' / d$

$$\min_{w, b} \frac{1}{2} \|w\|^2 \quad \text{subject to} \quad y_i (x_i^T w + b) \geq 1, \text{ where } d = \frac{1}{\|w\|}$$

Convex optimization problem with quadratic objective function and linear constraints.
Multiply constraint equations by positive Lagrange multipliers and subtract them from the objective function:

\[
L_p = \frac{1}{2} \| w \|^2 - \sum_{i=1}^{N} \alpha_i \left[ y_i (x_i^T w + b) - 1 \right]
\]

Min. \( L_p \) w.r.t. \( w \) and \( b \) and max. w.r.t. \( \alpha_i \), subject to \( \alpha_i \geq 0 \).

Set derivatives \( \delta L_p / \delta w \) and \( \delta L_p / \delta b \) to zero and max. w.r.t. \( \alpha_i \):

\[
w = \sum_{i=1}^{N} \alpha_i y_i x_i, \quad \sum_{i=1}^{N} \alpha_i y_i = 0.
\]

Substituting in \( L_p \) we get the so called Wolfe dual:

\[
\text{max. } L_D = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{N} \alpha_i \alpha_k y_i y_k x_i^T x_k
\]

subject to \( \alpha_i \geq 0 \), \( \sum_{i=1}^{N} \alpha_i y_i = 0 \)

\[
\text{solve for } \alpha_i \text{ then compute } w = \sum \alpha_i y_i x_i \text{ and } b \text{ from } \alpha_i \left[ y_i (x_i^T w + b) - 1 \right] = 0
\]
SVM—Primal vs. Dual (LS)

Primal:

\[
\min_{w,b} \frac{\|w\|}{2} \quad \text{subject to: } y_i (x_i^T w + b) \geq 1
\]

Dual:

\[
\max \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{i=1}^{N} \alpha_i \alpha_k y_i y_k x_i^T x_k \quad \text{subject to: } \alpha_i \geq 0, \quad \sum_{i=1}^{N} \alpha_i y_i = 0
\]

The primal has a dense inequality constraint for every point in the training set. The dual has a single dense equality constraint and a set of box constraints which makes it easier to solve than the primal.
SVM—Optimality Conditions (LS)

Optimality conditions for the linearly separable data:
\[
\sum_{i=1}^{N} \alpha_i y_i = 0, \quad \alpha_i \geq 0 \ \forall i, \quad y_i (x_i^T w + b) - 1 \geq 0 \ \forall i
\]

\[
w = \sum_{i=1}^{N} \alpha_i y_i x_i \quad \Rightarrow \quad f(x) = \sum_{i=1}^{N} \alpha_i y_i x_i^T x + b
\]

\[
\alpha_i (y_i (x_i^T w + b) - 1) = 0 \ \forall i \quad \Rightarrow \quad \alpha_i = 0 \text{ for points which are not on the boundary of the margin.}
\]

Points with \(\alpha_i > 0\) are support vectors.
SVM—Linear, non-separable (LNS)

\[ f(x) = 1 \]
\[ f(x) = 0 \]
\[ f(x) = -1 \]

\[ 2d = M \]

\[ \xi_i \]

\[
\min_{\mathbf{w}, b} \frac{1}{2} \| \mathbf{w} \|^2 + C \sum_{i=1}^{N} \xi_i \quad \text{subject to:}
\]
\[ y_i (\mathbf{x}_i^T \mathbf{w} + b) \geq 1 - \xi_i \quad \forall i, \quad \xi_i > 0 \quad \forall i \]

\( \xi_i \) are called slack variables

\( C \) constant, penalizes errors
SVM—Dual (LNS)

Same procedure as in separable case

\[ \text{max. } L_D = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{N} \alpha_i \alpha_k y_i y_k x_i^T x_k \]

subject to \( 0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{N} \alpha_i y_i = 0 \)

solve for \( \alpha_i \) then

compute \( w = \sum \alpha_i y_i x_i \) and

\( b \) from \( y_i (x_i^T w + b) - 1 = 0 \)

for any sample \( x_i \) for which \( 0 < \alpha_i < C \)
SVM—Optimality Conditions (LNS)

\[ f(x) = \sum_{i=1}^{N} \alpha_i y_i x_i^T x + b \]

\[ \alpha_i = 0 \Rightarrow \xi_i = 0, \quad y_i f(x_i) > 1 \]
\[ 0 < \alpha_i < C \Rightarrow \xi_i = 0, \quad y_i f(x_i) = 1 \]
\[ \alpha_i = C \Rightarrow \xi_i \geq 0, \quad y_i f(x_i) \leq 1 \]
SVM—Non-linear (NL)

Non-linear mapping:
\[ x' = \Phi(x) \]

Project into feature space, apply SVM procedure
SVM—Kernel Trick

\[
\text{max. } \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{N} \alpha_i \alpha_k y_i y_k x_i^T x'_k \\
\text{subject to } 0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{N} \alpha_i y_i = 0
\]

Only the inner product of the samples appears in the objective function. If we can write: \( K(x_i, x_k) = x_i^T x'_k \)
we can avoid any computations in the feature space.

The solution \( f(x') = x'^T w + b \) can be written as:

\[
f(x) = \sum_{i=1}^{N} \alpha_i y_i \Phi(x)^T \Phi(x_i) + b = \sum_{i=1}^{N} \alpha_i y_i K(x, x_i) + b
\]
using \( w = \sum \alpha_i y_i x'_i \).
SVM—Kernels

When is a Kernel \( K(u, v) \) an inner product in a Hilbert space?

\[
K(u, v) = \sum_n \lambda_n \phi_n(u)\phi_n(v)
\]

with positive coefficients \( \lambda_n \)

if for any \( g(u) \in L_2 \)

\[
\int K(u, v)g(u)\overline{g}(v)du dv \geq 0 \quad \text{Mercer's condition}
\]

Some examples of commonly used kernels:

Linear kernel: \( u^T v \)

Polynomial kernel: \( (1 + u^T v)^d \)

Gaussian kernel (RBF): \( \exp(-\|u - v\|^2) \), shift invar.

MLP: \( \tanh(u^T v - \theta) \)
SVM—Polynomial Kernel

Polynomial second degree kernel

\[ K(u, v) = (1 + u^T v)^2, \quad u, v \in \mathbb{R}^2 \]

\[ = 1 + (u_1 v_1)^2 + (u_2 v_2)^2 + 2u_1 v_1 u_2 v_2 + u_1 v_1 + u_2 v_2 \]

\[ = (1, u_1^2, u_2^2, \sqrt{2}u_1 u_2, u_1, u_2)(1, v_1^2, v_2^2, \sqrt{2}v_1 v_2, v_1, v_2)^T \]

\[ \Phi(x) = (1, x_1^2, x_2^2, \sqrt{2}x_1 x_2, x_1, x_2)^T \]

Shift invariant kernel \( K(u, v) = K(u - v) \)

defined on \( L^2([0, T]^d) \) can be written as the Fourier series of \( K \):

\[ f(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \lambda_k e^{\frac{j2\pi kt}{T}}, \quad f(t - t_0) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \lambda_k e^{\frac{j2\pi kt}{T}} e^{-\frac{j2\pi kt_0}{T}} \]

\[ K(u - v) = \sum_{k=0}^{\infty} \lambda_k e^{j2\pi k u} e^{-j2\pi k v} \quad \forall k \in \mathbb{Z}([-\infty, \infty]^d) \]
Are the $\alpha_i$ in the solution unique? No

$$f(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i y_i \mathbf{x}_i^T \mathbf{x} + b$$

$\mathbf{w} = (1, 0)$

two solutions:

$\alpha = (0.25, 0.25, 0.25, 0.25)$

$\alpha = (0.5, 0.5, 0, 0)$

constraints: $\alpha_i \geq 0$, $\sum_{i=1}^{N} \alpha_i y_i = 0$ are satisfied

More important: is the solution $f(\mathbf{x})$ unique?

Yes, the solution is unique and global

if the objective function is strictly convex.
SVM—Multiclass

Bottom-Up 1vs1

A or B or C or D

A or B

C or D

A

B

C

D

Training: \( k \frac{(k-1)}{2} \)

Classification: \( k-1 \)

1 vs. All

A / B,C,D

B / A,C,D

C / A,B,D

D / A,B,C

Training: \( k \)

Classification: \( k \)
SVM—Choosing the Kernel

How to choose the kernel?

Linear SVMs are simple to compute, fast at runtime but often not sufficient for complex tasks.

SVM with Gaussian kernels showed excellent performance in many applications (after some tuning of sigma). Slow at run-time.

Polynomial with 2\textsuperscript{nd} are commonly used in computer vision applications. Good trade off between classification performance computational complexity.
SVM—Example

Face detection with linear and 2\textsuperscript{nd} degree polyn. SVM & LDA

(CMU Testset 1, 127 images, 479 faces, 56,774.966 windows, res 19x19, pos 2429, neg 19932)
SVM—Choosing $C$

How to choose the $C$-value?

$$\min_{w,b} \frac{1}{2} \|w\|^2 + C \sum_{i=1}^{N} \xi_i$$

$C$-value penalizes points within the margin.

Large $C$-value can lead to poor generalization performance (over-fitting).

From own experience in **object detection** tasks: Find a kernel and $C$-values which give you zero errors on the training set.
In computer vision applications fast classification is usually more important than fast training.

Two ways of computing the decision function $f(x)$:

a) $\mathbf{w}^T \Phi(x) + b$

b) $\sum_{i=1}^{N} \alpha_i y_i K(x, x_i) + b$

Which one is faster?

- For a linear kernel a)
- For a polynomial 2nd degree kernel:

Multiplications for a): $G_{\Phi,\text{poly}2} = (n + 2)n$, where $n$ is dim. of $x$

Multiplications for b): $G_{K,\text{poly}2} = (n + 2)s$, where $s$ is nb. of sv's

- Gaussian kernel: only b) since dim. of $\Phi(x)$ is $\infty$. 
Learning Theory—Problem Formulation

From a given set of training examples \( \{x_i, y_i\} \) learn the mapping \( x \rightarrow y \). The learning machine is defined by a set of possible mappings \( x \rightarrow f(x, \alpha) \) where \( \alpha \) is the adjustable parameter of \( f \).

The goal is to minimize the expected risk \( R \):

\[
R(\alpha) = \int V(f(x, \alpha), y) \, dP(x, y)
\]

\( V \) is the loss function

\( P \) is the probability distribution function

We can't compute \( R(\alpha) \) since we don't know \( P(x, y) \)
Learning Theory – Empirical Risk Minimization

To solve the problem minimize the "empirical risk"

\[ R_{\text{emp}} \]

over the training set:

\[ R_{\text{emp}}(\alpha) = \frac{1}{N} \sum_{i=1}^{N} V(f(x_i, \alpha), y_i) \]

\( V \) is the loss function

Common loss functions:

\[ V(f(x), y) = (y - f(x))^2 \] least squares

\[ V(f(x), y) = (1 - yf(x))_+ \] hinge loss where \((x)_+ \equiv \max(x, 0)\)
Bound on the expected risk:

For a loss function with $0 \leq V(f(x), y) \leq 1$ with probability $1 - \eta$, $0 \leq \eta \leq 1$ the following bound holds:

$$R(\alpha) \leq R_{emp}(\alpha) + \sqrt{\frac{h \ln(2N/h) + h - \ln(\eta/4)}{N}}$$

$R_{emp}(\alpha)$ empirical risk

$N$ number of training examples

$h$ Vapnik Chervonenkis (VC) dimension

Bound is independent of the probability distribution $P(x, y)$.

Keep all parameters in the bound fixed except one:

$(1 - \eta) \uparrow$ bound $\uparrow$, $N \uparrow$ bound $\downarrow$, $h \uparrow$ bound $\uparrow$
The VC dimension is a property of the set of functions \( \{ f(\alpha) \} \). If for a set of \( N \) points labeled in all \( 2^N \) possible ways one can find an \( f \in \{ f(\alpha) \} \) which separates the points correctly one says that the set of points is shattered by \( \{ f(\alpha) \} \). The VC dimension is the maximum number of points that can be shattered by \( \{ f(\alpha) \} \).

The VC dimension of a functions \( f : w^T x + b = 0 \) in 2 dim:
The expected risk $E(R)$ for the optimal hyperplanes:

$$E(R) \leq \frac{E(D^2 / M^2)}{N}$$

where the expectation is over all training sets of size $N$.

'Algorithms that maximize the margin have better generalization performance.'
Most bounds on expected risk are very loose to compute instead:

**Cross Validation Error**
Error on a cross validation set which is different from the training set.

**Leave-one-out Error**
Leave one training example out of the training set, train classifier and test on the example which was left out. Do this for all examples. For SVMs upper bounded by the # of support vectors.
Regularization Theory

Given $N$ examples $(x_i, y_i), x \in \mathbb{R}^n, y \in \{0,1\}$ solve:

$$\min_{f \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} V(f(x_i), y_i) + \gamma \|f\|_K^2$$

where $\|f\|_K^2$ is the norm in a Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}$, with the reproducing kernel $K$, $\gamma$ is the regularization parameter.

$\gamma \|f\|_K^2$ can be interpreted as a smoothness constraint.

Under rather general conditions the solution can be written as:

$$f(x) = \sum_{i=1}^{N} c_i K(x, x_i)$$
Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}$

$f(x) = \langle \tilde{K}(x,y), f(y) \rangle_{\mathcal{H}}$

Positive numbers $\lambda_n$ and orthonormal set of functions $\phi_n(x)$, $\int \phi_n(x)\phi_m(x)dx = 0$ for $n \neq m$, and 1 otherwise:

$K(x,y) = \sum_n \lambda_n \phi_n(x)\phi_n(y)$, $\lambda_n$ are nonnegative eigenvalues of $K$

$f(x) = \sum_n a_n \phi_n(x)$, $a_n = \int f(x)\phi_n(x)dx$,

$\langle f(x), f(y) \rangle_{\mathcal{H}} = \sum_n \frac{a_n}{\sqrt{\lambda_n}} \frac{b_n}{\sqrt{\lambda_n}}$

$\|f(x)\|_{\mathcal{H}} = \langle f(x), f(x) \rangle_{\mathcal{H}} = \sum_n a_n^2 / \lambda_n$
Regularization—Simple Example of RKHS

Kernel is a one dimensional Gaussian with $\sigma = 1$:

$$K(x, y) = \exp(-(x - y)^2), \ x, y \ in \ [0,1]$$

write $K(x, y)$ as Fourier expansion using shift theorem:

$$K(x, y) = \sum_{n} \lambda_n \exp(j2\pi nx) \exp(-j2\pi ny) \quad \text{Period } T = 1$$

where $\lambda_n$ are the Fourier coeff. of $\exp(-x^2)$

$$\lambda_n = A \exp(-n^2 / 2)$$

$\lambda_n$ decreases with higher frequencies (increasing $n$).

This is a property of most kernels. The regularization term:

$$\|f(x)\|_H = \sum_{n} a_n^2 / \lambda_n, \text{ where } a_n \text{ are the Fourier coeff. of } f(x)$$

penalizes high freq. more than low freq. $\rightarrow$ smoothness!
Regularization—SVM

For the hinge loss function $V(f(x), y) = (1 - yf(x))^+$ it can be shown that the regularization problem is equivalent to the SVM problem:

$$\min_{f \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} (1 - y_i f(x_i))^+ + \lambda \|f\|^2_K$$

introducing slack variables $\xi_i = 1 - y_i f(x_i)$ we can rewrite:

$$\min_{f \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} \xi_i + \lambda \|f\|^2_K , \text{ subject to: } y_i f(x_i) \geq 1 - \xi_i, \text{ and } \xi_i \geq 0 \ \forall i$$

It can be shown that this is equivalent to the SVM problem (up to $b$):

**SVM:**

$$\min_{w, b} \frac{1}{2} \|w\|^2 + C \sum_{i=1}^{N} \xi_i \quad C = 1/(2\lambda N)$$

subject to: $y_i (x_i^T w + b) \geq 1 - \xi_i, \quad \xi_i \geq 0 \ \forall i$
SVM—Summary

- SVMs are maximum margin classifiers.
- Only training points close to the boundary (support vectors) occur in the SVM solution.
- The SVM problem is convex, the solution is global and unique.
- SVMs can handle non-separable data.
- Non-linear separation in the input space is possible by projecting the data into a feature space.
- All calculations can be done in the input space (kernel trick).
- SVMs are known to perform well in high dimensional problems with few examples.
- Depending on the kernel, SVMs can be slow during classification.
- SVMs are binary classifiers. Not efficient for problems with large number of classes.
Literature

T. Hastie, R. Tibshirani, J. Friedman: The Elements of Statistical Learning, Springer, 2001: 
*LDA, QDA, extensions to LDA, SVM & Regularization.*

C. Burges: A Tutorial on SVM for Pattern Recognition, 1999: *Learning Theory, SVM.*

R. Rifkin: Everything Old is New again: A fresh Look at Historical Approaches in Machine Learning, 2002: *SVM training, SVM multiclass.*

T. Evgeniou, M. Pontil, T. Poggio: Regularization Networks and SVMs, 1999: *SVM & Regularization.*

V. Vapnik: The Nature of Statistical Learning, 1995: *Statistical learning theory, SVM.*
Homework

Classification problem on the NIST handwritten digits data involving PCA, LDA and SVMs.

PCA code will be posted today