Given the differential equation: \( \frac{d^2 w}{dx^2} = \frac{Q_o}{EI} \cdot (L - x) \) on \( 0 \leq x \leq L \) where capitalized letters are constants. The boundary conditions are \( \frac{dw}{dx} \bigg|_{x=0} = 0 \) and \( w \bigg|_{x=0} = 0 \)

Find \( w(x) = ? \) \( \frac{dw}{dx} \bigg|_{x=L} = ? \) and \( w \bigg|_{x=L} = ? \)

From direct integration of the differential equation \( w(x) = \frac{Q_o}{EI} \cdot \left( \frac{x^2}{2} \cdot L - \frac{x^3}{6} \right) + c_1 \cdot x + c_2 \)

From the boundary conditions, the two constants of integration must be zero, so

\[
\begin{align*}
    w(x) &= \frac{Q_o}{EI} \cdot \left( \frac{x^2}{2} \cdot L - \frac{x^3}{6} \right) \\
    \frac{dw}{dx} &= \frac{Q_o}{EI} \cdot \left( xL - \frac{x^2}{2} \right)
\end{align*}
\]

Then, at \( x=L \), the deflection, \( w(L) = \Delta \), and the slope, \( \frac{dw}{dx} \bigg|_{x=L} = \phi(L) \) are

\[
\begin{align*}
    \Delta &= w(L) = \frac{Q_o \cdot L^3}{3EI} \\
    \phi(L) &= \frac{d}{dx}w \bigg|_{x=L} = \frac{Q_o \cdot L^2}{2EI}
\end{align*}
\]

What does all this have to do with the picture shown?

\( Q_o \) is the end load. \( w(x) \) is the deflection. The differential equation we start with is the “moment-curvature” relation. For small deflections and rotations, \( \frac{d^2 w}{dx^2} \) is the curvature (positive concave upwards). The right hand side is the bending moment which, by convention of 1.050, is taken as positive when the curvature is concave downwards. \( M_y = -Q_o \cdot (L - x) \)
Given the differential equation: \( \frac{d^2 w(x)}{dx^2} = \frac{-M_o}{EI} \) on \( 0 \leq x \leq L \) where all capitalized letters are constants. The boundary conditions are \( \frac{dw}{dx} \bigg|_{x=0} = 0 \) and \( w \big|_{x=0} = 0 \).

Find \( w(x) = ? \) \( \frac{dw}{dx} \bigg|_{x=L} = ? \) and \( w \big|_{x=L} = ? \)

As in the previous exercise, we find, after applying the boundary conditions

\[
w(x) = \frac{-M_o x^2}{2EI} \quad \text{and} \quad \frac{d}{dx} w(x) = \frac{-M_o x}{EI}
\]

Then, at \( x = L \), the deflection, \( w(L) = \Delta \), and the slope, \( \frac{dw}{dx} \bigg|_{x=L} = \phi(L) \) are

\[
\Delta = w(L) = \frac{-M_o L^2}{2EI} \quad \text{and} \quad \phi(L) = \frac{d}{dx} (w) \bigg|_{x=L} = \frac{-M_o L}{EI}
\]

What does all this have to do with the picture shown?

Here, \( M_o \) is the applied moment at the end of the beam. Within the beam, \( 0 < x < L \), the bending moment is constant and equal to the applied moment. It is positive according to our convention.

We now superimpose these two loading cases to solve a more general problem and one that is relevant to your design task.
For the cantilever under end load and end moment, the deflected shape might look as shown.

For this combined loading, we have:

\[
\frac{Q_o}{EI} \cdot \left( \frac{x^2}{2} - \frac{x^3}{6} \right) + \frac{-M_o x^2}{2EI} = \frac{d}{dx} \left( \frac{Q_o}{EI} \cdot \left( xL - \frac{x^2}{2} \right) + \frac{-M_o x}{EI} \right)
\]

At the end of the beam, we have:

\[
\Delta = \frac{Q_o \cdot L^3}{3EI} - \frac{M_o L^2}{2EI} \quad \text{and} \quad \phi(L) = \frac{Q_o \cdot L^2}{2EI} - \frac{M_o L}{EI}
\]

We want the solution for the special case when the slope at \( x = L \) is zero.

Setting \( \phi(L) = 0 \) gives the applied end moment in terms of the end force \( M_o \bigg|_{\phi(L) = 0} = \frac{Q_o \cdot L}{2} \)

So, in this case, the tip deflection is \( \Delta = \frac{Q_o \cdot L^3}{12EI} \) or, expressing \( Q_o \) in terms of the displacement,

\[
Q_o = K \cdot \Delta \quad \text{where the stiffness, } K, \text{ is } K = \frac{12 \cdot EI}{L^3}
\]
Engineering Beam Theory

Engineering beam theory shows that the most significant stress is the normal stress component on an “x face”; $\sigma_x$ in the example at the right. It is related to the applied loads by

$$\sigma_x = \frac{M_y \cdot z}{I}$$

where $z$ is the distance from the “neutral axis” which, for a doubly symmetric beam, is at the center of the cross-section and $I$ is the moment of inertia of the cross section.

$$I = \int A \cdot z^2 \, dA$$

For a rectangular cross-section of width $b$ and height $h$, this is $I = bh^3/12$

The applied loads come in through the bending moment $M_y$. The convention for positive shear and bending moment is shown in the figure.

The extensional strain, from the stress/strain relations is just $\varepsilon_x = \sigma_x/E$ where $E$ is the Elastic Modulus. In terms of the geometry of deformation, the extensional strain is given by $\varepsilon_x = z/R$ where $R$ is the radius of curvature of the neutral axis. $(1/R)$ is the curvature. The “bending stiffness” is defined as the product $EI$ as it appears in the “moment-curvature” relationship $M_y = (EI) \cdot \left(\frac{1}{R}\right)$ where the curvature, for small deflections, is related to the vertical displacement of the neutral axis by, $(1/R) = -\frac{d^2 w(x)}{dx^2}$

or $-\frac{M_y}{(EI)} = \frac{d^2 w(x)}{dx^2}$

An integration of the differential equation obtained from the moment curvature relation gives, for the case where the beam is loaded as shown, the mid-span deflection

$$w|_{\text{midspan}} = -\left(\frac{Pa}{24EI}\right) \cdot (3L^2 - 4a^2)$$