4.1 Notation and key definitions

4.1.1 Deformation

\[
\begin{align*}
(L + e)^2 &= (L \sin \theta + v)^2 + (L \cos \theta + u)^2 \\
(L^2 + 2Le + e^2) &= L^2 + 2eL \left(1 + \frac{e}{2L}\right) = L^2 + 2eL \text{ Assuming } e \ll L
\end{align*}
\]

Then

\[
L^2 + 2eL = L^2 s^2 + 2Lsv + v^2 + L^2 c^2 + 2Lcu + u^2
\]

where

\[
s = \sin \theta \text{ and } c = \cos \theta
\]

Combine terms

\[
L^2(s^2 + c^2) + 2Lsv + v^2 + 2Lcu + u^2 = L^2 + 2eL
\]

\[
e = cu + sv + \frac{1}{2L}(u^2 + v^2)
\]

\[
e = \alpha u + \frac{1}{2L}u^T u
\]

\[
\beta = \begin{bmatrix} \cos \psi & \sin \psi \end{bmatrix}
\]

\[
\cos \psi = \frac{L \cos \theta + u}{L + e} = \frac{\cos \theta + u}{1 + e/L} \approx \cos \theta + \frac{u}{L}
\]

\[
\cos \psi = \frac{L \sin \theta + v}{L + e} = \frac{\sin \theta + v}{1 + e/L} \approx \sin \theta + \frac{v}{L}
\]

Then

\[
\beta = \begin{bmatrix} \cos \theta + \frac{u}{L} & \sin \theta + \frac{v}{L} \end{bmatrix}
\]

\[
\beta = \alpha + \frac{1}{L}u^T
\]
4.1.2 Force

\[ F = \frac{AE}{L} e \]

\( \hat{F} \) has direction defined by \( \psi \)

Define

\[ P = \text{End action matrix} \]

\[ P = \begin{bmatrix} F \cos \psi \\ F \sin \psi \end{bmatrix} = F \beta^T \]

At positive end \( P|_{\text{+end}} = +P \)

At negative end \( P|_{\text{-end}} = -P \)

4.2 Incremental Equations

4.2.1 Deformation

Consider an incremental end displacement that changes \( u \) to \( u + \Delta u \). Find the corresponding change in elongation from \( e \) to \( e + \Delta e \).

\[ e = \alpha u + \frac{1}{2L} u^T y \]

\[ (e + \Delta e) = \alpha (u + \Delta u) + \frac{1}{2L} (u + \Delta u)^T (u + \Delta u) \]

\[ (e + \Delta e) = \alpha u + \alpha \Delta u + \frac{1}{2L} (u^T y + u^T \Delta u + \Delta u^T y + \Delta u^T \Delta u) \]

\[ \Delta e = (e + \Delta e) - e = \alpha \Delta u + \frac{1}{L} u^T \Delta u + \frac{1}{2L} \Delta u^T \Delta u \]

\[ \Delta e = d e + \frac{1}{2} d^2 e \]

\[ d e = \alpha \Delta u + \frac{1}{L} u^T \Delta u = \beta \Delta u \]

\[ d^2 e = \frac{1}{L} \Delta u^T \Delta u \]

If \( \Delta u \) is small in comparison to \( u \), it is reasonable to take \( \Delta e \approx de \)

Then \[ \boxed{\Delta e \approx \beta \Delta u} \]
4.2.2 Force

\[
\sigma = \sigma(\varepsilon)
\]

Non Linear Stress-Strain

\[
\sigma + \Delta \sigma = \sigma(\varepsilon + \Delta)
\]

From a Taylor Series expansion

\[
\Delta \sigma = \frac{\partial \sigma}{\partial \varepsilon} \Delta \varepsilon + \frac{1}{2} \frac{\partial^2 \sigma}{\partial \varepsilon^2} \Delta \varepsilon^2 + \ldots
\]

Note \( \frac{\partial \sigma}{\partial \varepsilon} = E_t \)

We want the change in stress due to the change in displacement

\[
\varepsilon = \frac{1}{L} e
\]

\[
\Delta \varepsilon = \frac{1}{L} \Delta e = \frac{1}{L} \left( de + \frac{1}{2} d^2 e \right)
\]

So

\[
\Delta \sigma = \frac{\partial \sigma}{\partial \varepsilon} \Delta \varepsilon + \left( \frac{1}{2} \frac{\partial^2 \sigma}{\partial \varepsilon^2} \Delta \varepsilon^2 \right) + \ldots
\]

\[
\Delta \sigma = d \sigma + \frac{1}{2} d^2 \sigma + \ldots
\]

\[
d \sigma = \frac{\partial \sigma}{\partial \varepsilon} \Delta \varepsilon = \frac{E_t}{L} de = \frac{E_t}{L} \beta \Delta u
\]

\[
d^2 \sigma = \frac{\partial \sigma}{\partial \varepsilon} d^2 e + \frac{\partial^2 \sigma}{\partial \varepsilon^2} (\Delta \varepsilon)^2 \sim \frac{\partial \sigma}{\partial \varepsilon} d^2 e + \frac{\partial^2 \sigma}{\partial \varepsilon^2} (de)^2
\]

The incremental forces are

\[
F = A \sigma
\]

\[
\Delta F = A \Delta \sigma = A d \sigma + \frac{A}{2} d^2 \sigma
\]

Set

\[
dF = A d \sigma = \frac{A E_t}{L} de = \frac{A E_t}{L} \beta \Delta u
\]

\[
d^2 F = \frac{A}{2} d^2 \sigma
\]

\[
A E_t \frac{L}{k_i} = k_i
\]

Then

\[
dF = k_i de = k_i \beta \Delta u
\]
4.2.3 End Actions

\[ P = F\beta^T \]

Consider the change in \( u \) to \( u + \Delta u \) and get the corresponding

\[
(P + \Delta P) = (F + \Delta F)(\beta^T + \Delta \beta^T)
\]

\[
(P + \Delta P) = F\beta^T + F\Delta \beta^T + \Delta F\beta^T + \Delta F\Delta \beta^T
\]

\[
\Delta P = (P + \Delta P) - P = F\Delta \beta^T + \Delta F\Delta \beta^T
\]

\[
\Delta P = dP + \frac{1}{2}d^2P + \ldots
\]

\[
dP = Fd\beta^T + dF\beta^T
\]

\[
\beta(u + \Delta u) = \beta + \Delta \beta = \beta + \frac{\partial \beta}{\partial u} \Delta u^T + \frac{1}{2} \frac{\partial^2 \beta}{\partial^2 u} \Delta u^T \Delta u
\]

\[
d\beta = \frac{\partial \beta}{\partial u} \Delta u^T = \left( \frac{\partial}{\partial u} \left( \alpha + \frac{1}{L} u^T \right) \right) \Delta u^T
\]

\[
d\beta = \frac{1}{L} \Delta u^T \quad d\beta^T = \frac{1}{L} \Delta u
\]

So

\[
dP = k_T \beta^T \Delta u + \frac{F}{L} \Delta u
\]

Can write as

\[
dP = k_T \Delta u
\]

where

\[ k_T = \text{Tangent member stiffness matrix} \]

\[
k_T = k_T \beta^T \beta + \frac{F}{L} I
\]

\[
k_T = \left( \frac{AE_T}{L} \right) \beta^T \beta + \frac{F}{L} I
\]

where

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{Identity matrix} \]

So

\[ \left( \frac{AE_T}{L} \right) \beta^T \beta \] is the pseudo linear stiffness based on the deformed position defined by \( \beta \) or \( \alpha \).

and

\[ \frac{F}{L} I \] is the “geometric” stiffness due to initial forces.
4.2.4 Illustration - Geometric stiffness associated with an initial force

![Diagram of geometric stiffness](image)

\[
\Delta u = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}
\]

Geometric term of \( dP = \frac{F}{L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \)

\[
dP = \frac{F}{L} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}
\]

This can also be done by inspection

\[
dP = \begin{bmatrix} F\cos\Delta\theta - F \\ F\sin\Delta\theta \end{bmatrix}
\]

Assuming small deformations \((e \ll L)\)

\[
\cos\Delta\theta = \frac{L + \Delta u}{L + e} \approx 1 + \frac{\Delta u}{L}
\]

\[
\sin\Delta\theta = \frac{\Delta v}{L + e} \approx \frac{\Delta v}{L}
\]

Therefore

\[
dP = \begin{bmatrix} F + \frac{F\Delta u}{L} - F \\ \frac{F\Delta v}{L} \end{bmatrix} = \frac{F}{L} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}
\]
4.3 System of Incremental Equilibrium Equations

\( P_E = \) External load vector
\( P_{int} = \) Nodal load vector due to end actions

For Equilibrium

\[ P_E + (-P_{int}) = 0 \]
\[ P_E = P_{int} \]

Let \( u^* \) be an estimate of displacement due to \( P_E \)

\[ P_{int}(u^*) = \text{Load vector (end actions) due to } u^* = P^*_{int} \]

Equilibrium imbalance = \( P_E - P^*_{int} \)

Let \( \Delta u \) be an estimate of the “correction” required to satisfy equilibrium.

The correction is found with: 
\[ k^* \Delta u = P_E - P^*_{int} \]
where \( k^* \) is a “predictor” stiffness matrix.

Once \( \Delta u \) is found, update \( P^*_{int} \) and \( k^* \) and use to determine another correction.

This process is repeated until the equilibrium error satisfies the specified convergence criteria.
Various choices for $k^*$

1) $k^* = k_{linear}$; the initial stiffness matrix

Successive iterations use the initial stiffness

Can also start with an initial displacement

2) $k^* = k_t$; the instantaneous tangent stiffness matrix

Update $k_t$ at each iteration. Requires less iteration but is computationally more expensive.
Example using $k^* = k_t$

\[ P_E = P \quad P_{init} = 2F\sin\theta \]

Using symmetry

Find the tangent stiffness of the system

\[ \beta = \alpha + \frac{1}{L}u^T = \alpha + \frac{1}{L}[u \ v] \]

\[ u = 0 \]

\[ \alpha = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \]

\[ \beta = \begin{bmatrix} \cos\theta \frac{h}{L} \end{bmatrix} + \frac{1}{L} \begin{bmatrix} 0 \ v \end{bmatrix} = \begin{bmatrix} \cos\theta \frac{h}{L} + \frac{v}{L} \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \end{bmatrix} \]

\[ dP = k\beta^T\beta\Delta u + \frac{F}{L}\Delta u \]

\[ \Delta u = \begin{bmatrix} 0 \\ \Delta v \end{bmatrix} \]

\[ dP = k\begin{bmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{bmatrix} \begin{bmatrix} 0 \\ \Delta v \end{bmatrix} + \frac{F}{L} \begin{bmatrix} 0 \\ \Delta v \end{bmatrix} \]

Set

\[ dP = \begin{bmatrix} 0 \\ \Delta P/2 \end{bmatrix} \]

So

\[ k\sin^2\psi\Delta v + \frac{F}{L}\Delta v = \frac{\Delta P}{2} \]

\[ 2\left(k\left(\frac{h}{L} + \frac{v}{L}\right)^2 + \frac{F}{L}\right)\Delta v = \Delta P \]

At each interval $k_f(v, F) = 2\left(k\left(\frac{h}{L} + \frac{v}{L}\right)^2 + \frac{F}{L}\right)$ must be updated.
For initial step take
\[ F = 0 \quad \rightarrow \quad k_{to} = 2kh^2 \]
\[ V = 0 \quad \rightarrow \quad k_{to} = \frac{2kL^2}{h^2} \]

Set
\[ k_{to}v_o = P \quad \rightarrow \quad v_o = \frac{P}{k_{to}} \]
and
\[ F_o = \frac{P}{2\sin\psi} \]

Update \( k_t \) to \( k_{t1} \) by including \( v_o \) and \( F_o \)
Get
\[ P_1 = k_{t1}v_o \]
\[ P_1 + \Delta P_1 = P \quad \rightarrow \quad \Delta P_1 = P - P_1 \]
\[ k_{t1}v_o + k_{t1}\Delta v = P \]
Then
\[ \Delta v_1 = \frac{P - k_{t1}v_1}{k_{t1}} \]

Repeat until \( \Delta v < v_{tol} \) or \( \Delta P_n < P_{tol} \)
4.4 Linearized Stability Analysis

4.4.1 Bifurcation

The system incremental equation expressed in terms of the instantaneous tangent stiffness matrix

\[ \Delta P = K_t \Delta u \]
\[ \Delta P = \text{load increment from the equilibrium position} \ (P^*, u^*) \]
\[ \Delta u = \text{first order estimate for displacement increment due to } \Delta P \]

Then \( \Delta u \) can be evaluated using

\[ \Delta u = (K_t)^{-1} \Delta P \]
and update \( P^*, u^*, K_t \) and \( \Delta P \)

Question: What happens if \( K_t \) becomes singular?

If \( |K_t| = 0 \), \( K_t \) is singular and there exists a non-trivial solution of \( K_t \Delta u = 0 \). This implies that there is more than one equilibrium position for a given load.

Suppose \( |K_t| = 0 \) for \( u^* \) and \( P^* \). Then, \( u^* + \Delta u \) is also a solution for the loading \( P^* \).

The existence of multiple equilibrium positions for a given loading is called “bifurcation.” In order to determine the structural behaviour in the neighborhood of a bifurcation point, we need to include second and higher order terms in the incremental equilibrium equations. Bifurcation is viewed as an intermediate state. The position is neither stable nor unstable. The terminology “neutral equilibrium” is used to interpret “bifurcation.”

4.4.2 Linearized Analysis

Consider the case where the equilibrium position (i.e. the displaced position) is very close to the initial unloaded position. In this case, it is reasonable to neglect the difference between the direction cosines for the initial and deformed configuration.

The expression for the member tangent stiffness reduces to

\[ K_t = k_t \beta^T \beta + \frac{F}{L^2} I = k_t \alpha^T \alpha + \frac{F}{L^2} I \]

When the material is linear elastic and the member is prismatic \( k_t = \frac{AE}{L} \)

This can be stated

\[ K_t = K_t + \frac{F}{L^2} I \]
Using this result, the system tangent stiffness matrix is expressed as
\[ K_t = K_l + K_g \]
where
\[ K_l = \text{linear system stiffness matrix} \]
\[ K_g = \text{“geometric” stiffness matrix containing the terms involving the member forces} \]
\[ K_g \] depends on the loading. As the loading is increased, \(|K_l|\) varies from its initial linear value \(|K_l|\). Depending on the loading and structural makeup, \(|K_l|\) may decrease with increasing load and eventually reach 0, at which point a “bifurcation” occurs. To evaluate whether this is possible, one evaluates \(|K_l + K_g|\) as a function of the loading.

### 4.4.2 Example 1

From equilibrium
\[ F_2 = -P \]
\[ k_{21} \equiv k_{l1} = \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix} \]
\[ k_{22} \equiv k_{l2} + k_{g2} = \begin{bmatrix} 0 & 0 \\ 0 & k_2 \end{bmatrix} + \frac{F_2}{L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
\[ k_{22} = \begin{bmatrix} 0 & 0 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} -P/L & 0 \\ 0 & -P/L \end{bmatrix} \]
\[ K_l = \begin{bmatrix} k_1 - P/L & 0 \\ 0 & k_2 - P/L \end{bmatrix} \]
\[ \text{Solve } \quad |K_l| = (k_1 - P/L)(k_2 - P/L) = 0 \]
\[ \text{Giving } \quad \text{For } P = k_1 L; \Delta u \text{ is arbitrary} \]
\[ \text{For } P = k_2 L; \Delta v \text{ is arbitrary} \]
4.4.3 Modified Geometric Stiffness Matrix

\[ \beta = \frac{1}{L + e} (\Delta x + \Delta u)^T \] (exact expression)

For (+) and (-) ends

\[ \Delta x = \text{difference in nodal coordinates} \]
\[ \Delta u = \text{difference in nodal displacements} \]

\[ \Delta x = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \quad \Delta u = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \]

Case 1 \( \Delta y \equiv 0 \) (Element – parallel to \( x \) direction)

\( \Delta u \) will be small wrt \( \Delta x \)

\[ \beta \equiv \frac{1}{L} \begin{bmatrix} \Delta x \\ \Delta y + \Delta v \end{bmatrix}^T \]

Then

\[ d\beta^T = \frac{\partial \beta}{\partial \Delta u} \Delta du = \frac{1}{L} \begin{bmatrix} 0 \\ \Delta dv \end{bmatrix}^T \]

and

\[ Fd\beta^T = \frac{F}{L} \begin{bmatrix} 0 \\ \Delta dv \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & F/L \end{bmatrix} \begin{bmatrix} \Delta du \\ \Delta dv \end{bmatrix} \]

Case 2 \( \Delta x \equiv 0 \) (Element – parallel to \( y \) direction)

\( \Delta v \) will be small wrt \( \Delta y \)

\[ \beta \equiv \frac{1}{L} \begin{bmatrix} \Delta x + \Delta u \\ \Delta y \end{bmatrix}^T \]

Then

\[ d\beta^T = \frac{1}{L} \begin{bmatrix} \Delta du \\ 0 \end{bmatrix}^T \]

and

\[ Fd\beta^T = \frac{F}{L} \begin{bmatrix} \Delta du \\ 0 \end{bmatrix}^T = \begin{bmatrix} F/L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta du \\ \Delta dv \end{bmatrix} \]
Generalizing

\[ k_g = \frac{F}{L} \]

Examine

\[ \alpha = \begin{bmatrix} \alpha_x & \alpha_y \end{bmatrix} \]

If \( \alpha_x \) close to 1, take \( E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \)

If \( \alpha_y \) close to 1, take \( E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \)

4.4.4 Example 2

\[ F_1 = -R \]
\[ F_2 = -P \]

\[ k_{i1} = \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{R}{L_1} \end{bmatrix} \]

\[ k_{i2} = \begin{bmatrix} 0 & 0 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} -\frac{P}{L_2} & 0 \\ 0 & 0 \end{bmatrix} \]

\[ K_1 = \begin{bmatrix} k_1 - \frac{P}{L_2} & 0 \\ 0 & k_2 - \frac{R}{L_1} \end{bmatrix} \]
4.4.5 Equilibrium Approach

Perturb structure from its initial equilibrium position, holding the loading constant. Apply equilibrium equations to the perturbed position. Determine whether a non-trivial solution for the “perturbed” displacement is possible. This approach is also seeking whether \(|K_t| = 0\). However, one does not formally establish \(K_t\).

Example 3

\[
\sum M_A = 0 = FL - PD
\]
\[
F = k\Delta u
\]

Then
\[
k\Delta uL = P\Delta u
\]
\[
P_{cr} = kL
\]

Example 4

Equilibrium Equations
\[
\sum M_{\text{top of } B} = P(\Delta u_2 - \Delta u_1) = F_2L_2
\]
\[
\sum M_A = P\Delta u_2 = F_2(L_1 + L_2) + F_1L_1
\]
Member Relations

\[ F_1 = k_1 \Delta u_1 \]
\[ F_2 = k_2 \Delta u_2 \]

Then

\[
\begin{bmatrix}
P & k_2L_2 - P \\
k_1L_1 & k_2(L_1 + L_2) - P
\end{bmatrix}
\begin{bmatrix}
\Delta u_1 \\
\Delta u_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Solve by setting determinant = 0

\[ P^2 - P(k_1L_1 + k_2(L_1 + L_2)) + k_1k_2L_1L_2 = 0 \]

If \( k_1 = k_2 \) and \( L_1 = L_2 \)

\[ \frac{P}{kL} = \frac{1}{2} \{ 3 \pm \sqrt{5} \} = 0.38, 2.62 \]

\[ \Delta u_2 = \Delta u_1 \left\{ \frac{P}{P - kL} \right\} \leftarrow \text{Mode Shapes} \]

4.4.6 Eigenvalue Approach

For the case where the loading is defined in terms of a single parameter, \( \lambda \), one can interpret linearized stability analysis as an eigenvalue problem.

Write

\[ F = \lambda \bar{F} \]
\[ k_i = k_i + \frac{F}{L} \]

Set

\[ \bar{k}_i = k_i - \lambda k_g \]

and the equilibrium equations can be written as

\[ k_i \Delta u = \lambda k_g \Delta u \]