

## Lecture 2: Stress

Geophysicists study phenomena such as seismicity, plate tectonics, and the slow flow of rocks and minerals called creep. One way they study these phenomena is by investigating the deformation and flow of Earth materials. The science of the deformation and flow of materials is called rheology.

The fundamental concepts of rheology are stress and strain. Stress is a dynamic quantity that expresses the magnitude of forces acting within or at the surface of continuous media and strain is a kinematic quantity that expresses deformation of the media. This lecture introduces the basic concepts of stress. It explains how to completely describe an object's state of stress in a 3D coordinate system and how to change the coordinate system for convenience. It also describes how to find the stress on an arbitrary plane within an object and how to determine if the stress on that plane will cause the object to break.

### 1. Forces—Surface Forces and Body Forces

Surface forces are different from body forces. For example, consider pushing a rock off the edge of a cliff. Pushing the rock involves a surface force that acts on an area bounding part of the rock. Dropping it off the cliff involves gravity, a body force that acts throughout the volume of the rock and depends on its mass.

### 2. Tractions

#### Motivation

If you push a balloon with your foot, it will slide across the ground. If you push a balloon with the same force using a needle, it will pop. The difference results from the fact that a needle and a foot apply force over a different surface area of the balloon. The needle applies force over a very small area and produces a large traction. The foot applies force over a much bigger area and produces a smaller traction.

#### Definition

A surface force  $\vec{F}$  that acts uniformly over planar surface of area  $A$  results in a traction  $\vec{T}$  of magnitude

$$\vec{T} = \frac{\vec{F}}{A}$$

The traction is a vector that has the same direction as  $\vec{F}$  and has units of pressure. In SI units, tractions are measured in Pascals (Pa).

$$\vec{T} = \frac{\vec{F}}{A} = \frac{\text{kg} \cdot \text{m}}{\text{m}^2 \cdot \text{s}^2} = \frac{\text{N}}{\text{m}^2} = \text{Pa}$$

Since tractions are vectors, they can be decomposed into normal and tangential vector components.  $\sigma_n$  and  $\tau$  denote the magnitude of these components.

$$\vec{T} = \vec{T}_{\text{tangential}} + \vec{T}_{\text{normal}}$$

$$\sigma_n = |\vec{T}_{\text{normal}}|$$

$$\tau = |\vec{T}_{\text{tangential}}|$$

Notes on the Orientation of Surfaces

Traction depends on the area of the surface over which it acts. Since area often changes with the orientation of the surface, the values of  $\sigma_n$  and  $\tau$  do not behave like vector components when the surface orientation changes. Consider the following figure:

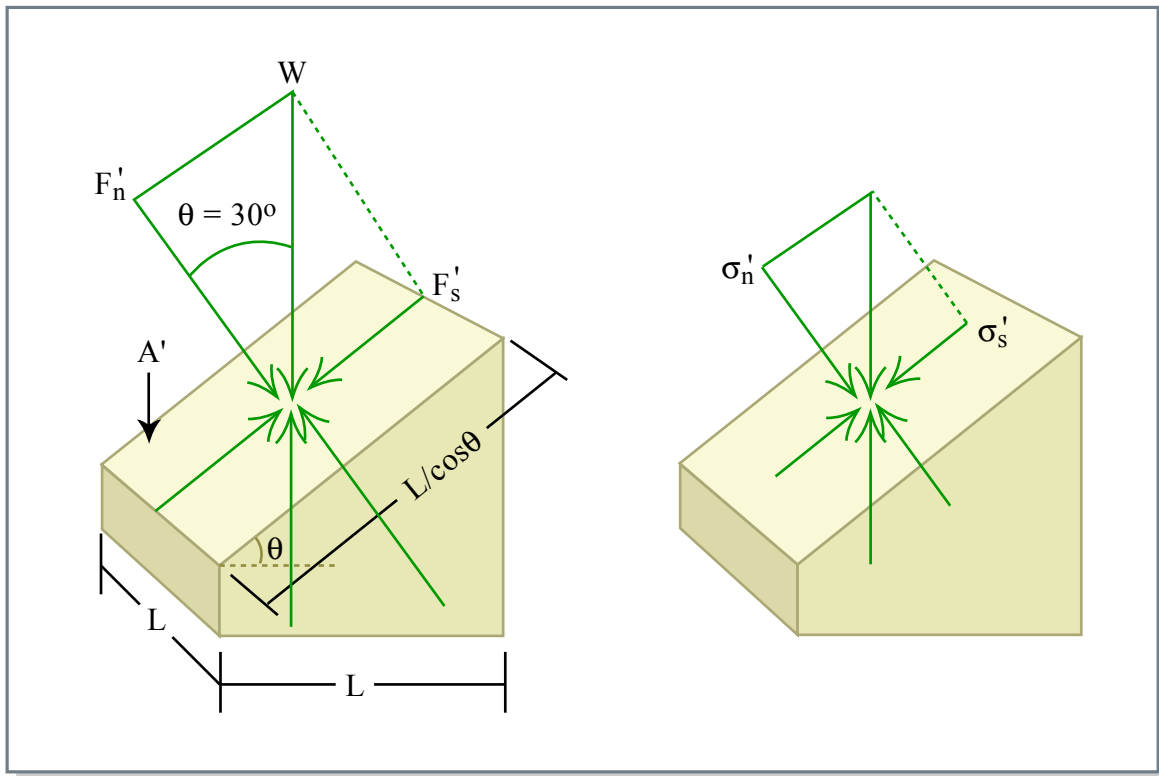


Figure 2.1

Figure by MIT OCW.

Compare how the normal and tangential forces change with the orientation of the surface with how the normal and tangential tractions change:

$$|\vec{F}_n| = W \cdot \cos \theta \qquad \sigma_n = W \cdot \cos^2 \theta$$

$$|\vec{F}_t| = W \cdot \sin \theta \qquad \tau = W \cdot \sin \theta \cdot \cos \theta$$

The decomposition of tractions into normal and shear components is useful in writing Admton's Law in terms of tractions. This new form of Admton's law is the link between stress and failure across a plane.

## Admanton's Law

Admanton's law relates a normal force and a tangential force by a proportionality constant called the coefficient of static friction  $f_s$ :

$$F_{\text{tangential}} = f_s \cdot F_{\text{normal}}$$

Dividing both sides of the equation by the area  $A$  over which the forces act leads to Admanton's law in terms of tractions:

$$\tau = f_s \cdot \sigma_n$$

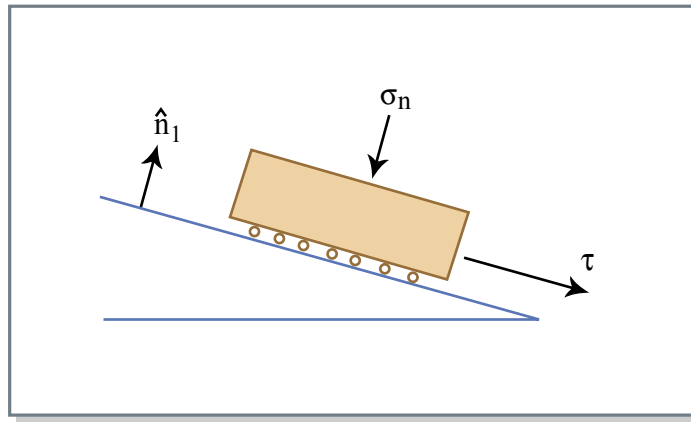


Figure 2.2

Figure by MIT OCW.

As shown in the picture, Admanton's law expresses a failure criterion. When the tangential force is larger than the product of the normal force and  $f_s$ , the block in the picture will slide down the incline. This failure criterion will help solve the problem of houses sliding into the ocean in Southern California. The problem in Southern California, though, is more difficult than the block problem because it is unclear which plane below the house has  $\sigma_n$  and  $\tau$  that will cause failure.

The rest of this lecture and the next one will address this difficulty. It will explore how to completely represent the state of stress below the house at any point, how to find the traction on an arbitrary plane below the house, and how to evaluate which one of those planes will fail.

### 3. Tractions at a point

#### Motivation

Tractions at a point are difficult to conceptualize because the area at a point over which a force acts is infinitesimal. Despite this difficulty, the concept of point tractions is extremely important because it allows one to find the traction on an arbitrary plane.

## Definition

Consider the free body in the picture below. Call the bounding surface  $S$ , the volume  $V$ , and a cutting plane through the body  $C$ . The normal to plane  $C$  is  $\hat{n}$ .  $P$  is a point on  $C$  and is surrounded by a small volume of  $C$  called  $\Delta C$ . Part 1 of the body exerts internal forces on Part 2 and gives rise to a force on  $\Delta C$  called  $\Delta f$ .

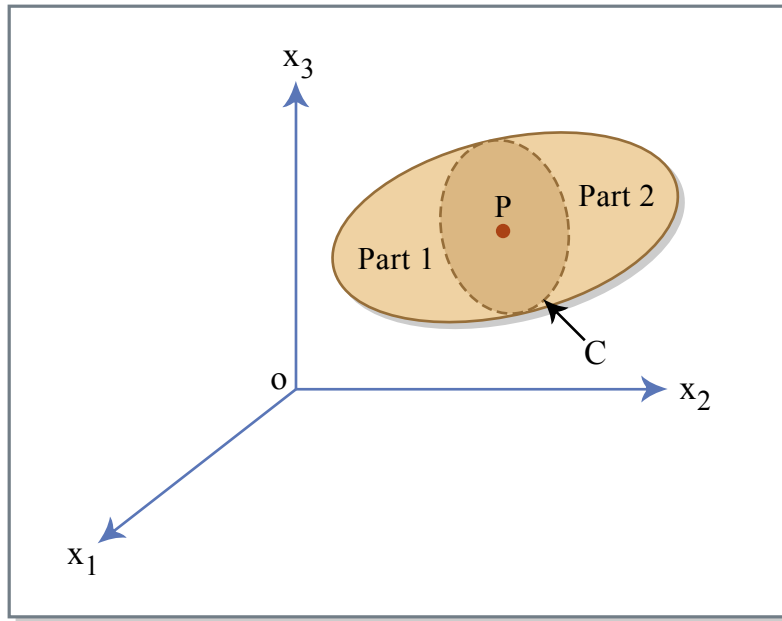


Figure 2.3

Figure by MIT OCW.

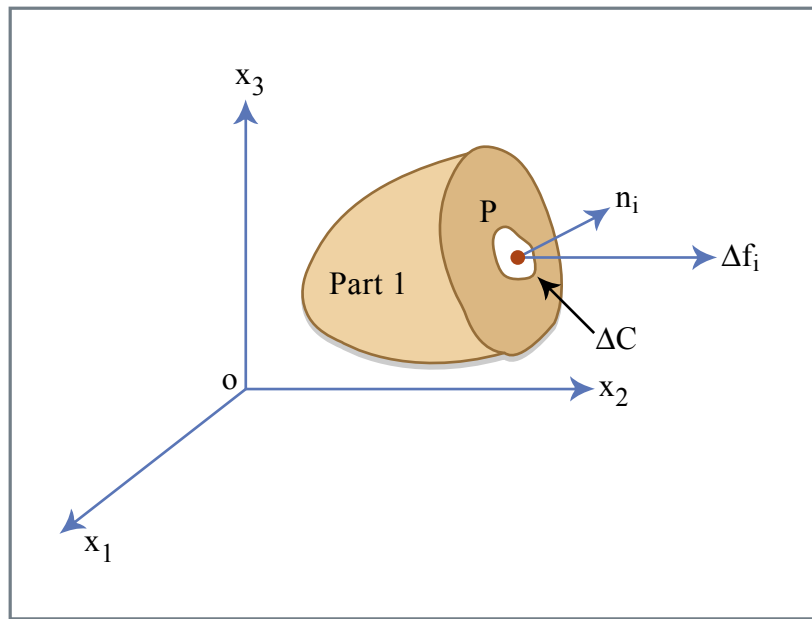


Figure 2.4

Figure by MIT OCW.

The Cauchy stress principle states that as the area around P shrinks to zero, the following limit holds

$$\lim_{\Delta C \rightarrow 0} \frac{\Delta f}{\Delta C} = T^{(\hat{n})}$$

$T^{(n)}$  is called the traction or stress vector at point P. The  $(\hat{n})$  is a reminder that this stress vector is defined only for a particular plane through P with normal vector  $(\hat{n})$ .

#### *Newton's Laws and the Traction at a Point*

Newton's third law says that the force exerted by Part 2 of the body on Part 1 is equal and opposite to the force exerted by Part 1 on Part 2. What about the point traction  $-T^{(-\hat{n})}$  exerted by Part 1 on Part 2?

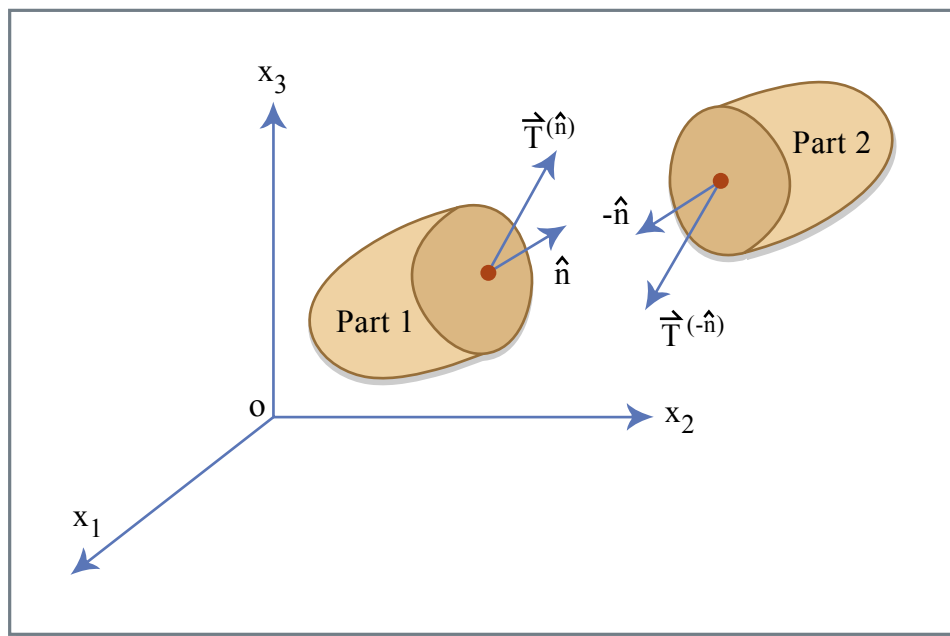


Figure 2.5

Figure by MIT OCW.

Applying Newton's second law to both parts of the body and the body as a whole reveals that  $-T^{(-\hat{n})}$  is equal to  $T^{(\hat{n})}$ . For a complete treatment of the problem see pages 50-51 in [Continuum Mechanics for Engineers](#) by Mase and Mase.

#### 4. The Complete Representation of Stress at a Point

##### Motivation

The last section provided the first step toward finding the stress on an arbitrary plane in continuous media. The next step is describing the stress tensor. The stress tensor is a representation of stress on three mutually perpendicular planes in a coordination system. It specifies the complete state of stress.

## Definition

### *Tensors*

Most physical quantities that are important in continuum mechanics like temperature, force, and stress can be represented by a tensor. Temperature can be specified by stating a single numerical value called a scalar and is called a zeroth-order tensor. A force, however, must be specified by stating both a magnitude and direction. It is an example of a first-order tensor. Specifying a stress is even more complicated and requires stating a magnitude and two directions—the direction of a force vector and the direction of the normal vector to the plane on which the force acts. Stresses are represented by second-order tensors.

### *The Stress Tensor*

Representing a force in three dimensions requires three numbers, each referenced to a coordinate axis. Representing the state of stress in three dimensions requires nine numbers, each referenced to a coordinate axis and a plane perpendicular to the coordinate axes. To understand what each of the nine numbers means, it is helpful to visualize an infinitesimally small cube in a continuous medium, oriented in a 3D coordinate system.

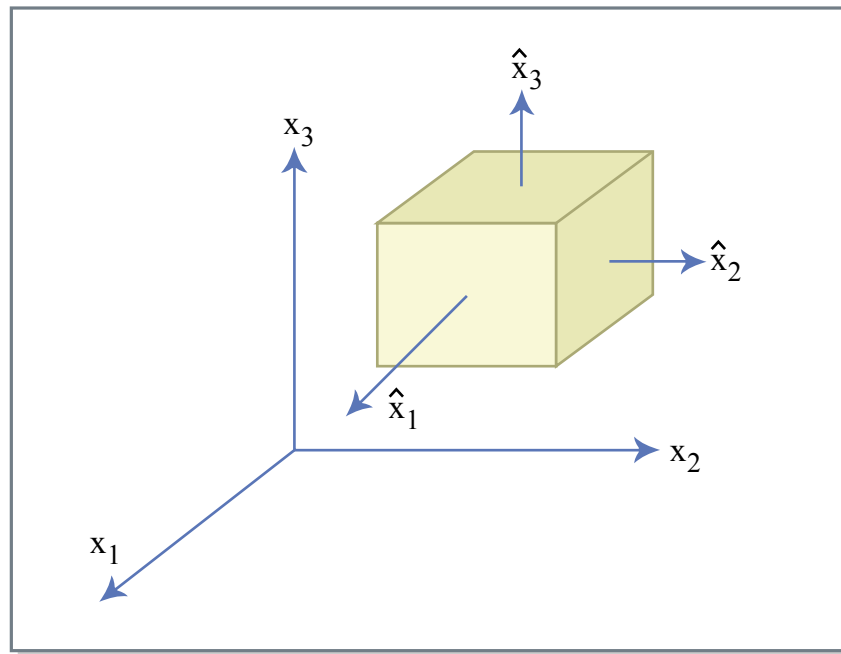


Figure 2.6

Figure by MIT OCW.

The coordinate system has axes  $x_1$ ,  $x_2$ ,  $x_3$  normal to the faces of the cube. The faces of the cube are defined by unit normal vectors  $\hat{x}_1$ ,  $\hat{x}_2$ ,  $\hat{x}_3$  that are positive when pointing out from the cube.

If there are uniform forces acting on each of the faces of the cube, then traction vectors can be defined on each face. Let the tractions on faces  $\hat{x}_1, \hat{x}_2, \hat{x}_3$  be called  $T^1, T^2, T^3$  respectively. These tractions are not necessarily normal to the faces.

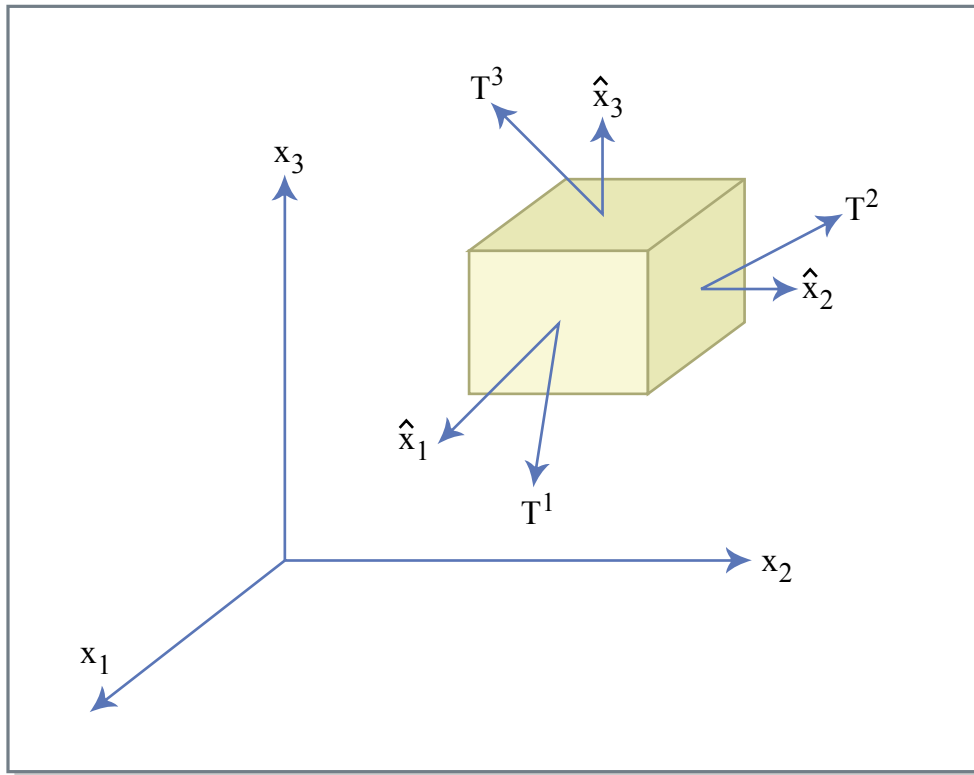


Figure 2.7

Figure by MIT OCW.

Tractions  $T^1, T^2, T^3$  are each defined by three components—one normal traction and two shear tractions. These components are labeled with two subscripts, as in  $\tau_{ij}$ . The first subscript  $i$  denotes the face of the cube on which the traction acts and the second subscript  $j$  denotes the direction of the traction.

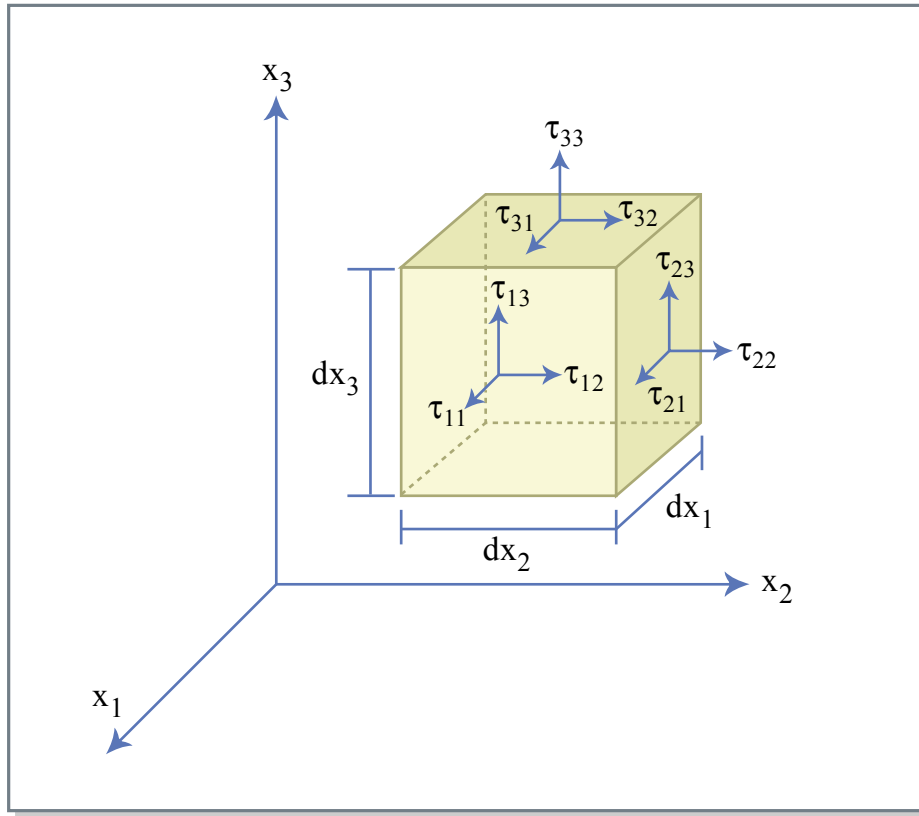


Figure 2.8

Figure by MIT OCW.

In vector notation, the tractions on the faces of the cube are written:

$$T^1 = \langle \tau_{11}, \tau_{12}, \tau_{13} \rangle$$

$$T^2 = \langle \tau_{21}, \tau_{22}, \tau_{23} \rangle$$

$$T^3 = \langle \tau_{31}, \tau_{32}, \tau_{33} \rangle$$

In matrix notation, the tractions are written:

$$\tau_{ij} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

This matrix is generally referred to as the stress tensor. It is the complete representation of stress at a point.

### *Features of the Stress Tensor*

The stress tensor is a symmetric tensor, meaning that  $\tau_{ij} = \tau_{ji}$ . As a result, the entire tensor may be specified with only six numbers instead of nine. For a proof, see Chapter 3.4 in Continuum Mechanics for Engineers by Mase and Mase.



## 5. The Cauchy Tetrahedron and Traction on Arbitrary Planes

### Application

This section shows that the traction vector at a point on an arbitrarily oriented plane can be found if  $T^1, T^2, T^3$  at that point are known.

### Argument

Apply Newton's second law to a free body in the shape of a tetrahedron and let the height of the tetrahedron shrink to zero.

### *Figures and Definitions*

Consider the tetrahedron below. The point O is the origin and the apices are labeled A, B, and C.

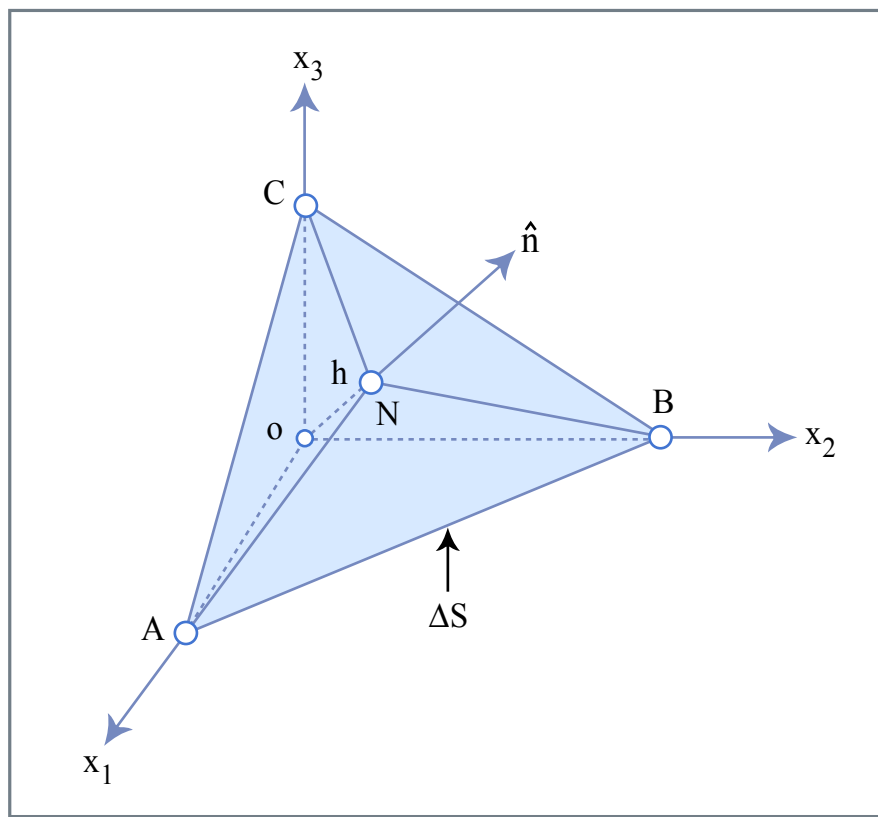


Figure 2.9

Figure by MIT OCW.

The relevant quantities are defined as follows:

$\rho$  = density

$\vec{F}_i$  = body force per unit mass in the  $i$  direction

$\vec{a}_i$  = acceleration in the  $i$  direction

$h$  = height of the tetrahedron, measured  $\perp$  to ABC

$\delta S$  = area of the oblique surface ABC

$\vec{T}_i$  = the component of the traction vector on the oblique surface in the  $i$  direction

The mass of the tetrahedron is  $\frac{1}{3}\rho h\delta S$ . The area of a face perpendicular to  $x_i$  is  $n_i\delta S$ .

### *Force Balance*

Consider the force balance in the  $i=1$  direction. Overbars denote values averaged over a surface or volume.

$$\vec{F}_1 = m\vec{a}_1$$
$$\vec{F}_1 \left( \frac{1}{3}\bar{\rho}h\delta S \right) + \bar{T}^1\delta S - \bar{\tau}_{11}(n_1\delta S) - \bar{\tau}_{21}(n_2\delta S) - \bar{\tau}_{31}(n_3\delta S) = \left( \frac{1}{3}\bar{\rho}h\delta S \right)\vec{a}$$

Divide both sides by  $\delta S$ .

$$\vec{F}_1 \left( \frac{1}{3}\bar{\rho}h \right) + \bar{T}^1 - \bar{\tau}_{11}(n_1) - \bar{\tau}_{21}(n_2) - \bar{\tau}_{31}(n_3) = \left( \frac{1}{3}\bar{\rho}h \right)\vec{a}$$

Allow  $h$  to approach zero in such a way that the surfaces and volume of the tetrahedron approach zero while the surfaces preserve their orientation. The body force and the mass both approach zero.

$$\bar{T}^1 = \bar{\tau}_{11}n_1 - \bar{\tau}_{21}n_2 - \bar{\tau}_{31}n_3$$

Performing the same force balance in the other two coordinate directions leads to expressions for the three traction components on an arbitrary plane.

$$T_1 = \tau_{11}n_1 + \tau_{21}n_2 + \tau_{31}n_3$$

$$T_2 = \tau_{12}n_1 + \tau_{22}n_2 + \tau_{32}n_3$$

$$T_3 = \tau_{13}n_1 + \tau_{23}n_2 + \tau_{33}n_3$$

The set of these three equations is called Cauchy's formula.

### *Different Notations*

1. A general equation for the explicit expressions above is given by:

$$T_i = \sum_{j=1}^3 \tau_{ji} n_j$$

2. Summation notation is a way of writing summations without the summation sign  $\Sigma$ . To use it, simply drop the  $\Sigma$  and sum over repeated indices. The equation in summation notation is given by:

$$T_i = \tau_{ji} n_j$$

3. The equation in matrix form is given by

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$