The Stream Function

For continuum mechanics in general and fluid mechanics specifically, a number of “laws” are expressed in terms of differential equations. For example,

1) Newton’s second law \( F = ma \) (general)
\[
\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \frac{Dv_i}{Dt}
\]

2) Rheology (constitutive equation) (Newtonian fluid)
\[
\sigma_{ij} = -p\delta_{ij} + 2\eta\dot{e}_{ij}
\]

3) Definition of strain rate (general)
\[
\dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
\]

4) Continuity (conservation of mass) (incompressible)
\[
\frac{\partial v_i}{\partial x_i} = 0
\]

These 4 coupled first order differential equations, plus boundary conditions, can be solved to determine fluid flow for a variety of interesting applications.

Alternatively, they can be combined to form a single fourth order differential equation.

For fluids, this fourth order equation often involves the stream function.

Consider a 2-D flow with velocities \( v_1, v_3 \) in the \( x_1, x_3 \) plane \( (v_2 = 0) \)
\[
\text{If } v_1 = -\frac{\partial \Psi}{\partial x_3}
\]
\[ v_3 = \frac{\partial \Psi}{\partial x_1} \Rightarrow \nabla \cdot v = \frac{\partial v_i}{\partial x_i} = -\frac{\partial^2 \Psi}{\partial x_i \partial x_j} + \frac{\partial^2 \Psi}{\partial x_j \partial x_i} = 0 \]

Incompressibility is automatically satisfied!

[In general, if \( \Psi = \nabla \times v \), \( \nabla \cdot \Psi = 0 \). Here \( \Psi = (0, \Psi, 0) \)]

Substituting into the (steady) Navier-Stokes equation

\[
-\frac{\partial p}{\partial x_1} - \eta \left( \frac{\partial^3 \Psi}{\partial x_1^2 \partial x_3} + \frac{\partial^3 \Psi}{\partial x_3^3} \right) + \rho f_1 = 0 \\
-\frac{\partial p}{\partial x_3} + \eta \left( \frac{\partial^3 \Psi}{\partial x_1^3} + \frac{\partial^3 \Psi}{\partial x_1 \partial x_3^2} \right) + \rho f_3 = 0
\]

Now take \( \frac{\partial}{\partial x_3} \) of first, \( \frac{\partial}{\partial x_1} \) of second

\[
-\frac{\partial^2 p}{\partial x_1 \partial x_3} - \eta \left( \frac{\partial^4 \Psi}{\partial x_1^2 \partial x_3^2} + \frac{\partial^4 \Psi}{\partial x_1 \partial x_3^4} \right) + \rho \frac{\partial f_1}{\partial x_3} = 0 \\
-\frac{\partial^2 p}{\partial x_1 \partial x_3} + \eta \left( \frac{\partial^4 \Psi}{\partial x_1^4} + \frac{\partial^4 \Psi}{\partial x_1^2 \partial x_3^2} \right) + \rho \frac{\partial f_3}{\partial x_1} = 0
\]

Subtract:

\[
\eta \left( \frac{\partial^4 \Psi}{\partial x_1^4} + 2 \frac{\partial^4 \Psi}{\partial x_1^2 \partial x_3^2} + \frac{\partial^4 \Psi}{\partial x_3^4} \right) + \rho \left( \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) = 0 \\
\frac{\partial^4 \Psi}{\partial x_1^4} + 2 \frac{\partial^4 \Psi}{\partial x_1^2 \partial x_3^2} + \frac{\partial^4 \Psi}{\partial x_3^4} = \nabla^2 (\nabla^2 \Psi) = \nabla^4 \Psi
\]

\( \nabla^4 \) is called biharmonic operator.

For uniform or no \( f \): \( \nabla^4 \Psi = 0 \)

Advantages of using the biharmonic operator are

1. only one equation
2. efficient solution
Disadvantage: Loss of “physical insight”.

Physical Interpretation of Stream Function
Consider triangle APB.

For incompressible fluid,

\[ \text{flux}_{AP} + \text{flux}_{BP} + \text{flux}_{AB} = 0 \]
\[ -v_3 \delta x_1 + v_1 \delta x_3 + \text{flux}_{AB} = 0 \]
\[ \text{flux}_{AB} = v_3 \delta x_1 - v_1 \delta x_3 = \frac{\partial \Psi}{\partial x_1} \delta x_1 + \frac{\partial \Psi}{\partial x_3} \delta x_3 = \partial \Psi \]

or

\[ \int_A^B d\Psi = \Psi_B - \Psi_A \]

Difference in \( \Psi \) represents the flux crossing the curve.

Solution of biharmonic

Polynomials (e.g., for Conette flow, \( \Psi = -\frac{v_0 x_3^2}{2h} \))

Separation of variables:

\[ \Psi = X(x)Z(z) \]

\[ \nabla^4 \Psi = 0 \Rightarrow X''''Z + 2X''Z'' + XZ''' = 0 \]
\[
\frac{X'''}{X} + 2 \frac{X'' Z'}{X Z} + \frac{Z'''}{Z} = 0
\]

Harmonic \( \Psi = \sin \frac{2\pi x}{\lambda} Z(z) \)

Solution: \( \Psi = [(A + Bz)\exp(\frac{2\pi z}{\lambda}) + (C + Dz)\exp(-\frac{2\pi z}{\lambda})] \sin(\frac{2\pi x}{\lambda}) \)

Physical boundary conditions: \( T_n = 0 \quad T_r = 0 \)

In \( x_1', x_3' \) coordinates, at \( x_3 = \xi(x_1) \):
\[
\sigma_{3,3}' = 0 \\
\sigma_{3,1}' = \sigma_{1,3}' = 0
\]

Have solution to biharmonic in terms of \( x_1, x_3 \) -- easily applied at \( x_3 = 0 \).

Need to take physical \( (x_1', x_3') \) boundary conditions and

1. rotate to \( x_1, x_3 \) space
2. Taylor's series expansion
3. subtract out hydrostatic reference state

Result (to first order in \( \xi/\lambda \) )
\[
\sigma = \begin{pmatrix}
? & ? & 0 \\
? & ? & 0 \\
0 & 0 & \rho g \xi
\end{pmatrix}
\]

4. solve biharmonic.
Postglacial Rebound

Decay of Boundary Undulations (1/2 space, uniform $\eta$)

- Assume uniform $\eta$
- Subtract out lithostatic pressure $P = p - \rho g x_3$
- Assume $\rho g$ uniform
- Use stream function $\Psi$

$$v_1 = -\frac{\partial \Psi}{\partial x_3} \quad v_3 = \frac{\partial \Psi}{\partial x_1}$$

$$\Rightarrow \nabla^4 \Psi = 0$$

Solution: $\Psi = \left[ (A + B k x_3) \exp(-k x_3) + (C + D k x_3) \exp(k x_3) \right] \cdot \sin k x_1$

Boundary conditions:

at $x_3 = 0$ (mathematical, not physical)

$$\sigma_{33} = \rho g \zeta$$

$$\sigma_{13} = 0 = \eta \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right)$$

at $x_3 \to \infty$, must be bounded
\( \Rightarrow C = D = 0 \)

In order that \( \sigma_{13} = 0 \) at \( x_3 = 0 \),

\[
- \frac{\partial^2 \Psi}{\partial x_3^2} + \frac{\partial^2 \Psi}{\partial x_i^2} = 0
\]

\( \Rightarrow B = A \)

or \( \Psi = A(1 + kx_3) \exp(-kx_3) \cdot \sin kx_i \)

Then

\[
\begin{align*}
v_1 &= Ak^2 x_3 \exp(-kx_3) \cdot \sin kx_i \\
v_3 &= Ak(1 + kx_3) \exp(-kx_3) \cdot \cos kx_i
\end{align*}
\]

at \( x_3 = 0 \) \( v_3 = \dot{\zeta} = Ak \cos(kx_i) \)

Now

\[
\sigma_{33} = -p + 2\eta \dot{\varepsilon}_{33} \\
\ddot{\varepsilon}_{33} = 0 \quad \text{at} \ x_3 = 0
\]

To get \( p \), use \( -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \rho x_i = 0 \)

for \( i = 1 \)

\( \Rightarrow -\frac{\partial p}{\partial x_1} + \eta(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_3^2}) = 0 \)

Substitute for \( v_1 \) and integrating \( \Rightarrow p_{|x_3=0} = 2\eta k^2 A \cos kx_1 \)

But \( p = -\rho g \zeta \Rightarrow A = -\frac{\rho g \zeta_0}{2k^2 \eta} \)

Or \( \dot{\zeta}_0 = -\frac{\rho g \zeta_0}{2k\eta} = -\frac{\rho g \lambda \zeta_0}{4\pi \eta} \)

Or \( \zeta_0 = \zeta_0 |_{t=0} \exp(-\frac{\rho g t}{2k\eta}) = \zeta_0 |_{t=0} \exp(-\frac{t}{\tau}) \)

where \( \tau = \frac{2k\eta}{\rho g} = \frac{4\pi \eta}{\rho g \lambda} \)

Solving for \( \eta : \quad \eta = \frac{\rho g \lambda \tau}{4\pi} \)

For curves shown,
\[ \tau : 5000 \text{ yr} \quad \lambda : 3000 \text{ km} \Rightarrow \eta : 10^{21} \text{ Pa} \]

Note: stream function \( \sim \exp(-kx_3) = \exp\left(-\frac{2\pi x_3}{\lambda}\right) \)

Falls off to \( \sim 1/e \) at \( x_3 : \frac{\lambda}{2\pi} \)

Senses to fairly great depth

\( \Rightarrow \) postglacial rebound doesn’t reveal the details of mantle viscosity structure, but only the gross structure.

Note: Behavior at Hudson Bay and Boston different:

**Hudson Bay**
- Continuous uplift

**Boston**
- Subsidence, then uplift

Is this consistent with uniform 1/2 space?

\[ \tau = \frac{4\pi\eta}{\rho g \lambda} \]

Decompose into Fourier components

\[ \Rightarrow ? \]

Details depend on geometry of ice load and elastic support of load.

Suppose we require faster relaxation for short \( \lambda \) than for long \( \lambda \).
How to get solution? What are the boundary conditions?