Flow in Porous Media

Problem of great economic importance (also scientific)
- hydrology (ground water migration, toxic waste)
- oil migration
- soil stability, fault mechanics (pore pressure)
- melt migration in mantle
- geysers and hot springs

Porous medium $\Rightarrow$ voids $\Rightarrow$ porosity $\phi$

$\phi \equiv$ volume fraction of voids

For example,
- Sand: $\phi \sim 40\%$
- Pumice: $\phi \sim 70\%$
- Oil shales: $\phi \sim 10-20\%$

If pore connected $\Rightarrow$ permeable

Pressure gradient $\Rightarrow$ flow
Darcy’s law $\Rightarrow \nu = -\frac{k}{\eta} \nabla p$

$\nu \equiv$ volumetric flow rate $\quad k \equiv$ permeability

We can use Poiseuille flow for simple geometries. For example, cubical matrix, circular tubes or pipes.
Figure 27.1. An idealized model of a porous medium. Circular tubes of diameter $\delta$ form a cubical matrix with dimensions $b$.

\[ \phi = \frac{12 \cdot \frac{1}{4} \cdot \pi \cdot \left( \frac{\delta}{2} \right)^2 \cdot b}{b^3} = \frac{3\pi \delta^2}{4b^2} \]

Consider $\frac{dp}{dx}$ (one direction only)

In each pipe (along $x$), $\bar{u} = -\frac{\delta^2}{32\eta} \frac{dp}{dx}$ [Poiseuille flow]

Darcy velocity: $v = \frac{4 \cdot \frac{1}{4} \cdot \bar{u} \cdot \pi \cdot \left( \frac{\delta}{2} \right)^2}{b^2} = \frac{\pi \delta^2 \bar{u}}{4b^2} = \frac{\phi \bar{u}}{3}$

\[ v = -\frac{b^2 \phi^2}{72\pi\eta} \frac{dp}{dx} \]

\[ \Rightarrow k = \frac{1}{72\pi} b^2 \phi^2 \]
Large $b \Rightarrow$ large $\nu$?  \[ b^2 = \frac{3\pi}{4} \frac{\delta^2}{\phi} \]

Large $\phi \Rightarrow$ large $\nu$?  \[ k = \frac{\pi}{128} \frac{\delta^4}{b^2} \]

Compare to cubes separated along faces (channel flow)

\[
\phi = \frac{6 \cdot \frac{1}{2} \cdot \delta b^2}{b^3} = 3 \frac{\delta}{b}
\]

Again, $\frac{dp}{dx}$ directed along one edge

\[
u = \frac{1}{2\eta} \frac{dp}{dx} \left( Z^2 - \left( \frac{\delta}{2} \right)^2 \right)
\]

\[
u = \frac{1}{2\eta} \frac{dp}{dx} \left( \frac{Z^3}{3} - \frac{\delta^2 Z}{2} \right)_{-\delta/2}^{\delta/2} = -\frac{5\delta^8}{24\eta} \frac{dp}{dx}
\]
Darcy velocity: 
\[ v = 2 \frac{b \delta}{b^2} \bar{u} = -\frac{5}{12} b \delta \frac{dp}{dx} = -\frac{5}{324} \frac{b^2 \phi^3}{\eta} \frac{dp}{dx} \]

\[ k = \frac{5b^2 \phi^3}{324} \]

\( k \) is different depending on \( \phi \).

\[ k = \frac{135 \delta^3}{324 b} \]

Clearly, porosity distribution is important.

Also -- more easily measured than figured out theoretically -- more complicated geometries \( \rightarrow \) numerical simulation.

Consider “Lawn Sprinkler” example – flow in unconfined aquifer.
Figure 27.4
Figure by MIT OCW.

\( h \equiv \text{"hydraulic head"} \)

\( u \rightarrow \text{Darcy velocity} \)

Dupuit approximation: \( \frac{dp}{dx} = \rho g \frac{\partial h}{\partial x} \)

For \( \frac{\partial h}{\partial x} = 1 \) flow is one-dimensional.

Darcy’s law: \( u = -\frac{k \rho g}{\eta} \frac{\partial h}{\partial x} \)

Conservation of mass: Assume no input

Flux \( Q = u(x)h(x) = -\frac{k \rho g}{\eta} h \frac{dh}{dx} = \text{const.} \)

\( \Rightarrow \) phreatic surface is a parabola

For \( h = h_0 \) at \( x = 0 \)

\[ h = \left( h_0^2 - \frac{2Q \eta x}{k \rho g} \right)^{1/2} \]
Suppose we have a porous dam of width \( w \). The relation between \( Q \), \( h_0 \) and \( h_1 \) is:

\[
Q = \frac{k \rho g}{2 \eta w} \left( h_0^2 - h_1^2 \right)
\]

or

\[
Q = \frac{k \rho g}{2 \eta w} \left[ (h_0 - h_1)(h_0 + h_1) \right]
\]

Figure 27.5. Unconfined flow through a porous dam. The Dupuit parabola AC is the solution if \((h_0-h_1)/h_0<<1\). The actual phreatic surface AB lies above the Dupuit parabola resulting in a seepage face BC.