Lecture 5: Changing Coordinate Systems and Mohr’s Circle

Lecture 2 explained that temperature is a zeroth-order tensor, force is a first-order tensor, and stress is a second-order tensor. The order of a tensor is called its rank and is defined by its law of transformation under a change of coordinates. This lecture explains the transformation laws for first- and second-order tensors, and uses these laws to derive a convenient representation of stress called Mohr’s circle.

1. Transforming Tensors into Different Coordinate Systems

Changing coordinate systems

In order to transform a tensor into a different coordinate system, one must first understand how to transform the coordinate system itself. Consider the following figure:

If \( \hat{x}_i \) and \( \hat{x}_j \) represent unit vectors that are the axes of two coordinate systems with the same origin, they are related by the equation

\[
\hat{x}_i' = \alpha_{ij} \hat{x}_j
\]

where \( \alpha_{ij} \) is the cosine of the angle between the primed axis \( \hat{x}_i' \) and the unprimed axis \( \hat{x}_j \). For example, \( \alpha_{12} \) is the cosine of the angle between \( \hat{x}_1' \) and \( \hat{x}_2 \). \( \alpha_{ij} \) represents a 9-component matrix called the transformation matrix. Unlike the stress tensor, it is not symmetric (\( \alpha_{ij} \neq \alpha_{ji} \)).
In matrix equations, the transformation law is written

\[
\begin{bmatrix}
    x_1' \\
    x_2' \\
    x_3'
\end{bmatrix} =
\begin{bmatrix}
    \alpha_{11} & \alpha_{12} & \alpha_{13} \\
    \alpha_{21} & \alpha_{22} & \alpha_{23} \\
    \alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
\]

The inverse transformation law is written

\[
\hat{x}_i = \alpha_{ji} \hat{x}_j
\]

**Example of coordinate transformations**

Consider the following transformation of coordinates:

![Figure 5.2](https://ocw.mit.edu/courses/mechanical-engineering/2-001d-materials-engineering-i-spring-2007/online-textbook/MIT2_001D_2007 lectures/5-2Coordinate变换-transform.png)

The transformation matrix is

\[
\alpha_{ij} = \begin{bmatrix}
    30^\circ & 60^\circ & 0 \\
    120^\circ & 30^\circ & 0 \\
    0 & 0 & 0
\end{bmatrix}
\]
The explicit transformation equations are

\[
\hat{x}_1' = \cos 30^\circ \hat{x}_1 + \cos 60^\circ \hat{x}_2 \\
\hat{x}_2' = \cos 120^\circ \hat{x}_1 + \cos 30^\circ \hat{x}_2
\]

Since \( \hat{x}_i' \) and \( \hat{x}_j \) are both unit length, these equations are easy to verify from the picture.

**First-order tensors**

First-order tensors or vectors have two components in 2D coordinates and three components in 3D coordinates. They transform according to the same laws as coordinate axes because coordinate axes are themselves vectors.

If \( u_j \) is a vector in the \( \hat{x}_j \) coordinate system and \( u_i' \) is a vector in the \( \hat{x}_i' \) coordinate system, then the following equations describe their transformation:

\[
u_i' = \alpha_{ij} u_j \\
u_i = \alpha_{ji} u_j'
\]

Note that \( \alpha_{ij} \) is positive if the angle is measured counterclockwise from \( \hat{x}_i' \) to \( \hat{x}_j \). It is negative if the angle is measured clockwise.

**Second-order tensors**

The transformation law for second-order tensors like stress and strain is more complicated than the transformation law for first-order tensors. It may be derived as follows:

a. Begin with the vector transformation of traction \( T_i \) to \( T_k' \):

\[
T_k' = \alpha_{ik} T_i
\]

b. Rewrite \( T_k' \) and \( T_i \) using Cauchy’s formulas:

\[
T_k' = \sigma'_{kl} n_l' \text{ and } T_i = \sigma_{ij} n_j
\]

Substitute Cauchy’s formulas into the original transformation equation:

\[
\sigma'_{kl} n_l' = \alpha_{ik} \sigma_{ij} n_j
\]
c. Transform the normal vector $n_j$ to $n_j'$ and substitute into the previous equation:

$$n_j = \alpha_j n_j'$$

$$\sigma'_{kl} n_j' = \alpha_{ki} \sigma_{ij} \alpha_{jl} n_j'$$

d. Cancel the $n_j'$ term on each side and group the $\alpha$s:

$$\sigma'_{kl} = \alpha_{ki} \alpha_{jl} \sigma_{ij}$$

Note that changing the position of the last $\alpha$ term changes the order of its subscripts.

In vector notation, the equation is

$$\underline{\underline{\sigma'}} = \underline{\underline{\alpha}} \underline{\underline{\alpha}}^T$$

where the double underbars denote second-rank tensors and the superscript $T$ denotes the transpose of matrix $\alpha$.

2. Mohr’s Circle

Motivation

Lecture II explained that an object resting on a slope will slide down when the shear traction on the slope is greater than or equal to the product of the normal traction and the coefficient of friction.

$$\tau = f_s \sigma_n$$

On a shallow slope, $\sigma_n$ is large and the object will not slide. On a steep slope, $\tau$ is large and the object will slide. For any plane with normal $\hat{n}$, we can calculate if the plane will fail if the stress tensor $\sigma_{ij}$ at the interface between the object and the slope is known. Calculate $\sigma_n$ and $\tau$ as follows:

**Vector and Tensor Notation**

$$\vec{T} = \underline{\underline{\sigma}} \hat{n}$$

$$\sigma_n = \vec{T} \cdot \hat{n}$$

$$\tau = \vec{T} - \vec{T} \cdot \hat{n}$$

**Summation Notation**

$$T_i = \sigma_{ij} n_j$$

$$\sigma_n = T_i n_i$$

$$\tau = T_i - T_i n_i$$

This method is straightforward but cumbersome. A different approach involves rotating the coordinate system such that $x_1'$ is along $\hat{n}$. In this case $\sigma_n$ and $\tau$ are much easier to derive:

$$\sigma_n = \sigma'_{11}$$

$$\tau = \sigma'_{12}$$
Deriving Mohr’s Circle

Mohr’s circle may be derived in two or three dimensions. This lecture explains the derivation in two dimensions because it is more straightforward and the results are easier to graph and understand. The derivation assumes that $x_1$, $x_2$, and $x_3$ are principle directions.

Consider the following figure in which the $x_i$ coordinate system is rotated clockwise about the $x_3$ axis to $x_i'$:

![Figure 5.3](https://ocw.mit.edu/courses/aeronautics-and-astronautics/12-818j-thick-skin-and-rapidly-deforming-flows-spring-2004/lecture_notes/5.3.png)

Figure 5.3
Figure by MIT OCW.

The rotation matrix is:

$$
\alpha_{ij} = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

The stress tensor $\sigma$ in the $x_i$ coordinate system is transformed to $\sigma'$ in the $x_i'$ coordinate system by the following equation:

$$
\sigma' = \alpha \sigma \alpha^T
$$
Use the double-angle identities for sine and cosine to simplify the expressions for the normal stress $\sigma_{11}'$ and the shear stress $\sigma_{12}'$ become in the new coordinate system:

\[
\sin 2\theta = 2 \sin \theta \cos \theta \\
\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta
\]

\[
\sigma_{11}' = \sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta \\
\sigma_{12}' = \tau = \frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta
\]

The expression for the normal stress and shear stress can be shown graphically in shear space:

![Mohr's circle](image)

**Figure 5.4**

Figure by MIT OCW.

This figure is called Mohr’s circle.
Interpreting Mohr’s circle

Mohr’s circle plots in stress space. Admonton’s law may also be plotted in stress space as a line with slope $f_s$. When this line and Mohr’s circle intersect, the criterion for failure across a plane is met.

Consider a common experiment in rock mechanics in which scientists apply a uniaxial stress $\sigma_2$ to a cylindrical sample confined by a uniform stress $\sigma_1$.

Admonton’s law and the Mohr’s circle that represents the state of stress are plotted below. The Mohr’s circle plots on the negative $\sigma_n$ axis because these notes follow the convention that compressive stresses are negative. $\sigma_1$ plots farther to the right because of the convention that $\sigma_1$ is greater than $\sigma_2$, and $\sigma_2$ is greater than $\sigma_3$. The line that represents Admonton’s law crosses the $\tau$ axis at $\sigma_0$, the shear strength of the rock.
As the uniaxial stress $\sigma_2$ is increased, the Mohr’s circle becomes larger. When the circle intersects the line of Admonton’s law, the rock breaks. The angle $2\theta$ at which the circle intersects the line is twice the angle between $\sigma_1$ and the normal vector to the failure plane.
Since most rocks have a coefficient of friction of about 0.6, the normal vector to the failure plane is typically 30° from the direction of the least compressive stress. Another way of saying this is that the failure plane is 30° from the direction of the most compressive stress.

<table>
<thead>
<tr>
<th>$f_s$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>45°</td>
</tr>
<tr>
<td>0.6</td>
<td>30°</td>
</tr>
<tr>
<td>1.0</td>
<td>23°</td>
</tr>
<tr>
<td>$\lim_{f_s \to \infty}$</td>
<td>0°</td>
</tr>
</tbody>
</table>

The above picture shows that the coefficient of friction determines the slope of Admonton’s law, and the slope of Admonton’s law determines $2\theta$. Consequently, predicting the failure plane of a rock only requires knowing the direction of the most compressive stress $\sigma_2$, the direction of the least compressive stress $\sigma_1$, and the coefficient of friction $f_s$. The table below lists the angle between $\sigma_1$ and the normal vector to the failure plane for different values of $f_s$.

Faulting

Faults are large-scale failure planes. Since failure planes are normal to the plane containing the least compressive and the most compressive stresses and are typically at 30° from the direction of the most compressive stress, different styles of faulting can be used to infer the directions of $\sigma_1$, $\sigma_2$, and $\sigma_3$.

The following diagram shows the deviatoric stresses associated with thrust faults, normal faults, and strike-slip faults. The pictures assume that $x_2$ is vertical and that $x_1$ and $x_3$ lie in the plane of the surface of the Earth.
Figure 5.8

Figure by MIT OCW.