12.540 Principles of the Global Positioning System
Lecture 03
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Review

• In last lecture we looked at conventional methods of measuring coordinates
• Triangulation, trilateration, and leveling
• Astronomic measurements using external bodies
• Gravity field enters in these determinations
Gravitational potential

- In spherical coordinates: need to solve

\[
\frac{1}{r} \frac{\partial^2}{\partial r^2} (rV) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0
\]

- This is Laplace’s equation in spherical coordinates
Solution to gravity potential

- The homogeneous form of this equation is a “classic” partial differential equation.
- In spherical coordinates solved by separation of variables, \( r = \text{radius}, \lambda = \text{longitude} \) and \( \theta = \text{co-latitude} \)

\[
V(r, \theta, \lambda) = R(r)g(\theta)h(\lambda)
\]
Solution in spherical coordinates

• The radial dependence of form \( r^n \) or \( r^{-n} \) depending on whether inside or outside body. \( N \) is an integer
• Longitude dependence is \( \sin(m\lambda) \) and \( \cos(m\lambda) \) where \( m \) is an integer
• The colatitude dependence is more difficult to solve
Colatitude dependence

• Solution for colatitude function generates Legendre polynomials and associated functions.

• The polynomials occur when $m=0$ in $\lambda$ dependence. $t=\cos(\theta)$

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$$
Legendre Functions

Low order functions. Arbitrary n values are generated by recursive algorithms.

\[ P_0(t) = 1 \]
\[ P_1(t) = t \]
\[ P_2(t) = \frac{1}{2} (3t^2 - 1) \]
\[ P_3(t) = \frac{1}{2} (5t^3 - 3t) \]
\[ P_4(t) = \frac{1}{8} (35t^4 - 30t^2 + 3) \]
Associated Legendre Functions

• The associated functions satisfy the following equation

\[ P_{nm}(t) = (-1)^m (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_n(t) \]

• The formula for the polynomials, Rodrigues’ formula, can be substituted
Associated functions

\[ P_{00}(t) = 1 \]
\[ P_{10}(t) = t \]
\[ P_{11}(t) = -(1 - t^2)^{1/2} \]
\[ P_{20}(t) = \frac{1}{2} (3t^2 - 1) \]
\[ P_{21}(t) = -3t(1 - t^2)^{1/2} \]
\[ P_{22}(t) = 3(1 - t^2) \]

- \( P_{nm}(t) \): \( n \) is called degree; \( m \) is order
- \( m \leq n \). In some areas, \( m \) can be negative. In gravity formulations \( m \geq 0 \)

http://mathworld.wolfram.com/LegendrePolynomial.html
Orthogonality conditions

- The Legendre polynomials and functions are orthogonal:

\[ \int_{-1}^{1} P_n'(t) P_n(t) dt = \frac{2}{2n+1} \delta_{nn} \]

\[ \int_{-1}^{1} P_{n'm}(t) P_{nm}(t) dt = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nn} \]
Examples from Matlab

• Matlab/Harmonics.m is a small matlab program to plots the associated functions and polynomials
• Uses Matlab function: Legendre
“Sectoral Harmonics” $m=n$
Normalized Sectoral harmonics: Degrees 2-5, order $m=n$

\[ \sqrt{\frac{2}{2m+1}} \frac{(n+m)!}{(n-m)!} \]
Surface harmonics

• To represent field on surface of sphere; surface harmonics are often used

\[ Y_{nm}(\theta, \lambda) = \sqrt{\frac{2m+1}{4\pi}} \frac{(n-m)!}{(n+m)!} P_{nm}^{m}(\theta)e^{im\lambda} \]

• Be cautious of normalization. This is only one of many normalizations

• Complex notation simple way of writing \( \cos(m\lambda) \) and \( \sin(m\lambda) \)
Surface harmonics

Code to generate figure on web site

Zonal ---- Tesserals ---------------------------Sectorial
Gravitational potential

- The gravitational potential is given by:

\[ V = \iiint \frac{G\rho}{r} dV \]

- Where \( \rho \) is density,
- \( G \) is Gravitational constant \( 6.6732 \times 10^{-11} \) m\(^3\)kg\(^{-1}\)s\(^{-2}\) (N m\(^2\)kg\(^{-2}\))
- \( r \) is distance
- The gradient of the potential is the gravitational acceleration
Spherical Harmonic Expansion

• The Gravitational potential can be written as a series expansion

\[
V = -\frac{GM}{r} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\cos \theta)[C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda)]
\]

• \(C_{nm}\) and \(S_{nm}\) are called Stokes coefficients
Stokes coefficients

• The Cnm and Snm for the Earth’s potential field can be obtained in a variety of ways.
• One fundamental way is that $1/r$ expands as:

$$
\frac{1}{r} = \sum_{n=0}^{\infty} \frac{d'^n}{d^{n+1}} P_n(\cos \gamma)
$$

• Where $d'$ is the distance to $dM$ and $d$ is the distance to the external point, $\gamma$ is the angle between the two vectors (figure next slide)
1/r expansion

• $P_n(\cos \gamma)$ can be expanded in associated functions as function of $\theta, \lambda$
Computing Stoke coefficients

• Substituting the expression for $1/r$ and converting $\gamma$ to co-latitude and longitude dependence yields:

$$P_n(\gamma) = \frac{4\pi}{2n+1} \sum_{m=0}^{n} Y_{nm}^*(\theta', \lambda') Y_{nm}(\theta, \lambda)$$

$$V = \iiint \frac{G dM}{r} = 4\pi \iiint \frac{dM}{2n+1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{d^n}{d^{n+1}} Y_{nm}^*(\theta', \lambda') Y_{nm}(\theta, \lambda)$$

The integral and summation can be reversed yielding integrals for the Cnm and Snm Stokes coefficients.
Low degree Stokes coefficients

- By substituting into the previous equation we obtain:

\[ C_{10} = GM \iiint z'dM \quad C_{11} = GM \iiint x'dM \]
\[ S_{11} = GM \iiint y'dM \]

\[ C_{20} = \frac{GM}{2} \iiint 2z^2 - x^2 - y^2 dM \]
\[ C_{21} = GM \iiint xzdM \quad S_{21} = GM \iiint yzdM \]
\[ C_{22} = \frac{GM}{4} \iiint x^2 - y^2 dM \quad S_{22} = \frac{GM}{2} \iiint xydM \]
Moments of Inertia

• Equation for moments of inertia are:

\[
I = \begin{bmatrix}
\int \int \int y^2 + z^2 \, dM & \int \int \int xy \, dM & \int \int \int xz \, dM \\
\int \int \int xy \, dM & \int \int \int z^2 + x^2 \, dM & \int \int \int yz \, dM \\
\int \int \int xz \, dM & \int \int \int yz \, dM & \int \int \int x^2 + y^2 \, dM
\end{bmatrix}
\]

• The diagonal elements in increasing magnitude are often labeled A, B and C with A and B very close in value (sometimes simply A and C are used)
Relationship between moments of inertia and Stokes coefficients

• With a little bit of algebra it is easy to show that:

\[
C_{20} = GM\left(\frac{A + B}{2} - C\right)
\]

\[
C_{22} = \frac{1}{4} GM(B - A)
\]

\[
S_{22} = \frac{1}{2} GMI_{12}
\]

\[
C_{21} \quad S_{21} \text{ are related to } I_{13} \text{ and } I_{23}
\]
Spherical harmonics

- The Stokes coefficients can be written as volume integrals
- $C_{00} = 1$ if mass is correct
- $C_{10}, C_{11}, S_{11} = 0$ if origin at center of mass
- $C_{21}$ and $S_{21} = 0$ if Z-axis along maximum moment of inertia
Global coordinate systems

• If the gravity field is expanded in spherical harmonics then the coordinate system can be realized by adopting a frame in which certain Stokes coefficients are zero.

• What about before gravity field was well known?
Summary

• Examined the spherical harmonic expansion of the Earth’s potential field.

• Low order harmonic coefficients set the coordinate.
  – Degree 1 = 0, Center of mass system;
  – Degree 2 give moments of inertia and the orientation can be set from the directions of the maximum (and minimum) moments of inertia. (Again these coefficients are computed in one frame and the coefficients tell us how to transform into frame with specific definition.) Not actually done in practice.

• Next we look in more detail into how coordinate systems are actually realized.