Slowly varying media: Ray theory

Suppose the medium is not homogeneous (gravity waves impinging on a beach, i.e. a varying depth). Then a pure plane wave whose properties are constant in space and time is not a proper description of the wave field.

However, if the changes in the background occur on scales that are long and slow compared to the wavelength and period of the wave, a plane wave solution may be locally appropriate. (Fig. 2.1) This means: $\lambda << L_m$ where $L_m$ is the length scale over which the medium changes. Consider the local plane wave

$$\phi(\vec{x},t) = a(\vec{x},t)e^{i\theta(\vec{x},t)}$$

$a$ varies on the scale $L_m$ while $\theta$ varies on the scale $\lambda$.

$$\frac{\partial \theta}{\partial x_i} = 0(\frac{1}{\lambda}); \quad \frac{1}{a} \frac{\partial a}{\partial x_i} = 0(\frac{1}{L_m})$$

$$\Rightarrow \nabla \phi = ae^{i\theta} \cdot \nabla \theta + \theta \left( \frac{\lambda}{L_m} \right)$$

with

$$\theta = \vec{k} \cdot \vec{x} - wt$$

Define the local wavenumber and the local frequency as:

$$\vec{k} = \nabla \theta \bigg|_t \quad \omega = -\frac{\partial \theta}{\partial t} \bigg|_x$$

From these definitions it follows that:

$$\nabla \times \vec{k} = 0 \quad \text{the local wave number is irrotational.}$$

Conservation of crests in a slowly varying medium.

Suppose we go from point A to point B over the curve $C_1$. 

1
slowly varying wave fronts

The number of wave crests we pass along $C_1$ is

$$n_{c_1} = \frac{1}{2\pi} \oint_{A} B \cdot k \, d\mathbf{s} = \frac{1}{2\pi} \int_{C_1} \mathbf{k} \cdot d\mathbf{s}$$

The number of wave crests we pass along $C_2$ is

$$n_{c_2} = \frac{1}{2\pi} \oint_{A} B \cdot k \, d\mathbf{s} = \frac{1}{2\pi} \int_{C_2} \mathbf{k} \cdot d\mathbf{s}$$

Before for plane waves $\omega = \Omega(k)$ only, now $\omega = \Omega(k, \mathbf{x}, t)$.

As $(\omega, k)$ are slowly varying functions of space/time, the dispersion relation is explicitly dependent on space/time. Now we can introduce the group velocity in another way

$$\frac{\partial \omega}{\partial t} |_{\mathbf{x}} = \frac{\partial \Omega}{\partial k_i} |_{k, \mathbf{x}, t} \frac{\partial k_i}{\partial t} |_{\mathbf{x}} + \sum_{i} c_{gi} \frac{\partial k_i}{\partial t} |_{\mathbf{x}}$$

Where we use the summation convention over repeated indices,

and $c_{gi} = \frac{\partial \Omega}{\partial k_i}$ by definition $i = 1, 2, 3 = x, y, z$
\[ \tilde{c}_g = \nabla \tilde{k} \Omega \quad \text{group velocity} \]

The difference is:

\[
n_{c_1} - n_{c_2} = \frac{1}{\pi} \left[ \int_{c_1} \left( \int \tilde{k} \cdot d\tilde{s} \right) - \int_{c_2} \left( \int \tilde{k} \cdot d\tilde{s} \right) \right] = \frac{1}{2\pi} \int_{\text{total}} \tilde{k} \cdot d\tilde{s} = \oint_{A} \nabla \times \tilde{k} \cdot \hat{n} dA \equiv 0
\]

\[ \hat{n} = \text{unit vector normal to } C \]

We have used Stokes theorem relating the line integral of the tangential component of \( \tilde{k} \) to the area integral of its curl over the area bounded by the closed contour \( C \). The increase of phase is the same on \( C_1 \) and \( C_2 \). This means the number of crests along \( C_1 \) is the same as the number of crests along \( C_2 \), that is the number of crests inside the area \( A \) is conserved. Crests are neither created nor destroyed inside \( A \). The crests have no ends, so the number of crests within a wave group will be the same for all time. This is obviously true only for slowly varying plane waves.

From the definition of \( \tilde{k} \) and \( \omega \) it follows:

\[
\frac{\partial \tilde{k}}{\partial t} + \nabla \omega = 0 \quad (1)
\]

We have seen that the number of crests we cross from \( A \) to \( B \) is the same along any path connecting \( A \) and \( B \). Then:

\[
n = \frac{1}{2\pi} \int_{A} B \tilde{k} \cdot d\tilde{s}
\]

\[
\frac{\partial n}{\partial t} = \frac{1}{2\pi} \int_{A} B \frac{\partial \tilde{k}}{\partial t} \cdot d\tilde{s} = -\frac{1}{2\pi} \int_{A} \nabla \omega \cdot d\tilde{s} = \frac{1}{2\pi} \left( \omega(A) - \omega(B) \right)
\]

This says that the rate of change of the number of wave crests between \( A \) and \( B \) is equal to the frequency of crest inflow at \( A \) minus the frequency crest outflow at \( B \).
Crests are neither created nor destroyed in the smoothly varying function $\phi$. The number in any local region increases or decreases solely due to the arrival of pre-existing crests at $A$, not to the creation or destruction of existing crests.

We now introduce the dynamics by asserting that the wavenumber and frequency must be related by a dispersion relation in the same way as for a plane wave.

Since by eq. (1)

$$\frac{\partial k_i}{\partial t} = -\frac{\partial \omega}{\partial x_i}$$

we have

$$\frac{\partial \omega}{\partial t} = \frac{\partial \Omega}{\partial t} - c_g \frac{\partial \omega}{\partial x_i}$$
or

$$\frac{\partial \omega}{\partial t} + c_g \cdot \nabla \omega = \frac{\partial \Omega}{\partial t} |_{k,\bar{x}} \tag{1}$$
equation for $\omega$

Similarly from (1)

$$\frac{\partial k_i}{\partial t} \bigg|_{x} + \frac{\partial \Omega}{\partial x_i} \bigg|_{k,t} + \frac{\partial \Omega}{\partial k_j} \bigg|_{k,\bar{t}} \frac{\partial k_j}{\partial x_i} = 0$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial k_j} \frac{\partial k_j}{\partial x_i} = -\frac{\partial \Omega}{\partial x_i}$$
or

$$\frac{\partial k_i}{\partial t} + c_g \cdot \nabla k = -\nabla \Omega |_{k,t} \tag{2}$$

The “ray equation” gives the velocity at which the wave packet, or wave group, moves:

$$c_g = \frac{dx}{dt} \quad \text{or} \quad c_{gx} = \frac{dx}{dt} ; \quad c_{gy} = \frac{dy}{dt}$$
in two dimensions. Then the ray path in the $(x,y)$ plane is
\[ \frac{dy}{dx} = \frac{c_{gy}}{c_{gx}} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \overline{c}_g \boldsymbol{\cdot} \nabla \]

\( \overline{c}_g = \frac{d\overline{x}}{dt} \quad (I) \)

\[ \frac{\partial \omega}{\partial t} + \overline{c}_g \boldsymbol{\cdot} \nabla \omega = \frac{\partial \Omega}{\partial t} |_{\overline{r},\overline{x}} \quad (II) \]

\[ \frac{\partial \overline{k}}{\partial t} + \overline{c}_g \boldsymbol{\cdot} \nabla \overline{k} = -\nabla \Omega |_{\overline{r},t} \quad (III) \]

\( \Omega = \Omega(\overline{k},\overline{x},t) \) has an explicit parametric dependence on \((\overline{x},t)\), for instance when waves enter in water of changing depth. The ray equations give the evolution of the local wavenumber \( \overline{k} \) and the local frequency \( \omega \) as we move along the ray, i.e. we move with the wave packet at the local group velocity \( \overline{c}_g \). Is a plane wave a particular solution of the ray theory formulation? Suppose the medium is homogeneous, no changes in \((\overline{x},t)\)

\( \omega = \Omega(\overline{k}) \) only

Solution: plane wave \( \phi = ae^{i(\overline{k} \cdot \overline{x} - \omega t)} \)

where \((\overline{k},\omega)\) do not change but are constant in space

Initial condition \( \phi(\overline{x}) = ae^{i\overline{k} \cdot \overline{x}} \) gives \( \overline{k}(t=0) \)

As \( \frac{\partial \overline{k}}{\partial x_2} \equiv 0 \) and \( \frac{\partial \Omega}{\partial x_i} \equiv 0 \)

The ray equation (III) gives

\[ \frac{\partial \overline{k}}{\partial t} = 0: \overline{k} \text{ never changes along the ray and remains equal to } \overline{k} (t=0). \]

\( \omega = \Omega(\overline{k}) \) gives \( \omega \) at \( t = 0 \)
As \( \frac{\partial \omega}{\partial x_1} = 0 \); \( \frac{\partial \Omega}{\partial t} = 0 \) \quad \text{eq. (II) gives}

\( \frac{\partial \omega}{\partial t} = 0 \quad \omega = \omega(t=0) \)

The frequency never changes along the ray. Thus the plane wave solution in a homogeneous medium is entirely consistent with the ray theory formulation.