4. Energy equation for surface gravity waves

Equations of Motion

\[ \rho \frac{d\mathbf{u}}{dt} = -\nabla p - g\rho \mathbf{k} \quad (1) \quad \rho = \text{constant} \]

\[ \nabla \cdot \mathbf{u} = 0 \quad (2) \quad D = \text{constant} \]

Multiply (1) by \( \mathbf{u} \)

\[ \left( \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right)_t + \mathbf{u} \cdot \nabla p + g\rho w = 0 \]

In the linearized case, at every level \( z \) \( w = \frac{\partial z}{\partial t} \) and

\[ \left[ \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + g\rho z \right]_t + \nabla \cdot (\rho \mathbf{u}) = 0 \]

or rate of change (kinetic + potential energy) + divergence (energy flux) = 0

If we integrate from \( z = -D \) to \( z = \eta \), we obtain the kinetic and potential energy and

energy flux per unit horizontal area:

\[ \frac{\partial}{\partial t} \left[ \frac{\eta}{2} \rho \mathbf{u} \cdot \mathbf{u}dz + \frac{1}{2} \rho \eta^2 \right] + \nabla H \cdot \mathbf{u} = \int_{-D}^{\eta} (\rho \mathbf{u} \cdot \mathbf{u} + g\rho z) dz = 0 \]

as \( p(\eta) = 0 \)

and \( w = 0 \) at \( z = -D \)

\[ \nabla H = i \frac{\partial}{\partial x} + J \frac{\partial}{\partial y} \]

\[ \int_{-D}^{\eta} g\rho z dz = \frac{1}{2} g\rho \frac{z^2}{2} \bigg|_{-D}^{\eta} = \frac{1}{2} g\rho (\eta^2 - D^2) \]
\[
\frac{\partial}{\partial t} [\text{KE} + \text{PE}] + \nabla H \cdot \mathbf{E}_{\text{flux}} = 0
\]

Rate of change = horizontal divergence of wave energy flux

Bar denotes the quantities per unit horizontal area

Notice:

1) In the expression for the integrated potential density:

\[
\frac{1}{2} \rho g (\eta^2 - D^2)
\]

we have neglected the term proportional to \(D^2\) as an irrelevant constant and \(\frac{\partial D^2}{\partial t} = 0\).

2) In the integral for the kinetic energy we can integrate only to \(z = 0\). In fact we are calculating energy to second order in the wave amplitude. To do this, for PE, we must integrate to \(\eta\) to obtain \(\eta^2 (\equiv a^2)\). In the KE, the integral to \(\eta\) would include a correction of \(0(u^2\eta)\equiv o(a^3)\), hence negligible. Let us now consider specifically the surface gravity wave field in one horizontal dimension (x,z,t):

\[
\eta = a \cos(kx - \omega t) \quad \omega^2 = gk \tanh(kD)
\]

\[
\phi = \frac{aw}{k \sinh(kD)} \cosh(kz + D) \cos(kx - \omega t)
\]

\[
p = -\rho g z + \frac{\rho \omega^2 a}{k \sinh(kD)} \cosh(kz + D) \cos(kx - \omega t)
\]

\[
u = \frac{a\omega}{\sinh(kD)} \cosh(kz + D) \cos(kx - \omega t)
\]

\[
w = \frac{a\omega}{\sinh(kD)} \sinh(kz + D) \sin(kx - \omega t)
\]
\[ \text{PE} = \frac{1}{2} \rho g a^2 \cos^2(kx - \omega t) \]

\[ \text{KE} = + \int_{-D}^{0} \rho \left( u^2 + w^2 \right) \frac{dz}{2} = \int_{-D}^{0} \rho \frac{a^2 \omega^2}{2} \left[ \frac{\cos^2(kx - \omega t) \cosh^2 k(z + D)}{\sinh^2 (kD)} \right] \left[ \frac{\sin^2(kx - \omega t) \sinh^2 k(z + D)}{\sinh^2 (kD)} \right] \]

Let us now average both quantities over a wave period, indicated by \(< >\)

\[ <\text{PE}> = \frac{1}{4} \rho g a^2 \]

\[ <\text{KE}> = \rho a^2 \omega^2 \int_{-D}^{0} \frac{1}{4} \frac{\cosh 2k(z + D)}{\sinh^2 (kD)} dz = \]

\[ = \rho \quad a^2 \omega^2 \quad \frac{1}{8} \frac{\sinh (2kD)}{k \sinh^2 (kD)} = \]

\[ = \rho a^2 g \tanh (kD) \frac{\sinh (kD) \cosh (kD)}{4 \sin^2 (kD)} = \frac{1}{4} \rho g a^2 \]

Averaged over a wave period

\[ <\text{PE}> = <\text{KE}> \quad \text{Equipartition of wave energy between potential and kinetic like in the oscillator problem.} \eta \text{ is a linear oscillator!} \]

And \[ <\text{E}_{\text{total}} > = <\text{KE} > + <\text{PE}> = \frac{\rho g a^2}{2} \]

If we now calculate the energy flux vector and average it over one wave period we get:

\[ <\text{E}_{\text{flux}} > = \int_{-D}^{0} (up)dz = \]

\[ = \frac{1}{2} \rho g a^2 \left( \frac{\omega^2}{gK} \coth (kD) \right) c \left[ \frac{1}{2} + \frac{KD}{\sinh (2kD)} \right] \]
But \( c_g = \frac{\partial \omega}{\partial k} = c \left[ \frac{1}{2} + \frac{kD}{\sinh(kD)} \right] \)

Thus the period average of the energy equation is:
\[
\frac{\partial}{\partial t} \langle E \rangle + \nabla H \cdot \left[ \tilde{c}_g \langle E \rangle \right] = 0
\]

Thus we have the important result that the energy in the wave propagates with the group velocity. If the medium is homogeneous, \( \tilde{c}_g = \frac{\partial \omega}{\partial k} \) only and we can write:
\[
\frac{\partial}{\partial t} \langle E \rangle + \tilde{c}_g \cdot \nabla H \langle E \rangle = 0
\]

For an observer moving horizontally with the group velocity the energy averaged over one phase of the wave is constant.

**Dispersion relationship for waves moving on a current**

Suppose I have a wave encountering a current \( \tilde{U}(x,y) \), the dispersion relationship is modified by the Doppler shift becoming
\[
\sigma = \tilde{k}(x,y) \cdot \tilde{U}(x,y) + \omega \quad \text{where} \quad \omega = \sqrt{gk \tanh(1k)}D \quad \text{is the intrinsic frequency}
\]

Consider in fact the 1-D example
\[
U = U(x) \quad \text{only. Then} \quad \sigma = kU + \omega.
\]