10. Unbounded domain - non-rotating reflection from a solid boundary

We consider the reflection from a solid boundary which is at some angle with the horizontal. Consider a two-dimensional solution

\[ e^{-i\omega t + ikx + imz} \]

aligning \( x \) with the horizontal wave vector \( k_H \)

satisfying

\[ w_{zz} - R^2 w_{xx} = 0. \]

with \( R^2 = \frac{N^2 - \omega^2}{\omega^2 - \nu^2} \) and \( m = \pm Rk \).

The lines of constant phase are \( \theta = kx + mz - \omega t = \text{constant} \) or:

\[ +kx \pm Rkz - \omega t = \text{constant} \]

that is

\[ x \pm Rz = \left( \frac{\omega}{k} \right) t = \text{constant} ; \]

energy propagates along the lines of constant phase

\( x \pm Rz = \text{constant} \) that is:

\[ z = \frac{1}{R} x \quad \text{(positive slope)} \quad z = -\frac{1}{R} x \quad \text{(negative slope)} \]

Figure by MIT OpenCourseWare.
These lines are the characteristics of the hyperbolic equation for \( w \), i.e.

\[
    w = f(x+Rz) + g(x-Rz)
\]

Consider first a wave incident and reflected at the horizontal boundary \( z = 0 \), i.e. the \( x \)-axis.

\( \vec{c}_{gi} \) downward: energy propagates along \( x+Rz=0 \). The incident wave number \( \vec{K}_i \) is perpendicular to \( \vec{c}_{gi} \) and upward, Energy is reflected along \( x-Rz \), upward \( \vec{c}_{gr} \). The reflected wave number \( \vec{K}_r \) is downward.

\( \omega = N\cos\theta \) is conserved in the reflection.

\( \theta \) is the angle of \( \vec{K} \) with the horizontal.

As \( \omega \) is determined only by \( \theta \), the angle to the horizontal, \( \vec{K}_i \) and \( \vec{K}_r \) must form equal angles \( \theta \) with the horizontal. In this particular case \( |\vec{K}_i|\cos\theta = |\vec{K}_r|\cos\theta \).

We can demonstrate that \( \omega \) is conserved as follows.

Let us consider the more general case of a wall inclined to the horizontal \( z = ax \) and let us
consider a 2-D problem. Then continuity is simply \( u_x + w_z = 0 \) and we can introduce a streamfunction \( \psi \)

\[
\begin{align*}
\psi &= -\frac{\partial \psi}{\partial z} \\
\psi &= +\frac{\partial \psi}{\partial x}
\end{align*}
\]

The incident wave, in terms of \( \psi \), is:

\[
\psi_I = \psi_{io} e^{i(k_i x + m_i z - \omega_i t)}
\]

and

\[
\psi_R = \psi_{ro} e^{i(k_r x + m_r z - \omega_r t)}
\]

The total wave field in the reflection is

\[
\psi_{Total} = \psi_I + \psi_R
\]

and on \( z = ax \) \( \psi_T = \text{constant} = 0 \) without loss of generality. Then

\[
\begin{align*}
\psi_{io} e^{i[(k_i + am_i)x - \omega_i t]} + \\
+ \psi_{ro} e^{i[(k_r + am_r)x - \omega_r t]} &\equiv 0
\end{align*}
\]

This is true only if

\[
\begin{align*}
\omega_i &= \omega_r \\
\omega_i &= \omega_r \\
k_i + am_i &= k_r + am_r \Rightarrow k_i + \tan \alpha m_i &= k_r + \tan \alpha m_r \\
as a = \tan \alpha \\
or \ k_i \cos \alpha + m_i \sin \alpha &= k_r \cos \alpha + m_r \sin \alpha \\
or \ \vec{K}_i \cdot \hat{i}_B = \vec{K}_r \cdot \hat{i}_B \quad \text{if} \quad \hat{i}_B \text{ is the unit vector along } z = ax
\end{align*}
\]

that is:

1. \( \omega \) is conserved in the reflection process

\( \Rightarrow \) the angle of \( \vec{K}_r \) and \( \vec{K}_i \) to the horizontal must have the same magnitude \( \theta \)
2. The component of $\vec{K}_i$ and $\vec{K}_r$ along the slope must be the same

Let us consider the geometry of the process:

$$x = z \tan \theta = R z \quad \tan \theta = R$$

$$\theta = \tan^{-1} R \quad \alpha = \tan^{-1} a$$

The projection of $\vec{K}_i, \vec{K}_r$ along the reflecting wall $z = ax$ must be equal:

$$|\vec{K}_i| \cos[\tan^{-1} R - \tan^{-1} a] = |\vec{K}_r| \cos[\tan^{-1} R + \tan^{-1} a]$$

We can evaluate this expression by geometry and the law of cosines:

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}$$
\[
\cos(\tan^{-1}R - \tan^{-1}a) = \frac{1 + R^2 + 1 + a^2 - (R - a)^2}{2\sqrt{1 + R^2 \sqrt{1 + a^2}}} = \frac{1 + aR}{\sqrt{1 + R^2 \sqrt{1 + a^2}}}
\]

\[
\cos(\tan^{-1}R + \tan^{-1}a) = \frac{1 + a^2 + 1 + R^2 - (R + a)^2}{2\sqrt{1 + R^2 \sqrt{1 + a^2}}} = \frac{1 - aR}{\sqrt{1 + R^2 \sqrt{1 + a^2}}}
\]
And the above expression becomes:

\[ | \tilde{K}_i | \frac{1 + aR}{\sqrt{1 + R^2} \sqrt{1 + a^2}} = | \tilde{K}_R | \frac{1 - aR}{\sqrt{1 + R^2} \sqrt{1 + a^2}} \]

or

\[ | \tilde{K}_R | = \frac{1 + aR}{1 - aR} | \tilde{K}_i | \]

or

\[ k_R = \frac{(1 + aR) k_i}{1 - aR} \]

\[ m_r = -\frac{(1 + aR) m_i}{1 - aR} \]

The reflected wave number

\[ | \tilde{K}_R | > | \tilde{K}_i | \]

\[ \Rightarrow \lambda_R < \lambda_i \]

The wavelength shortens as a consequence of the reflection process.

Consider now the changes in group velocity \( \tilde{c}_g \)

For the group velocity the component conserved is the component perpendicular to the wall as there cannot be an energy flux into the wall

\[ c_{g_i} |_{\perp \text{wall}} - c_{g_r} |_{\perp \text{wall}} = 0 \]

\[ c_{g_i} |_{\perp \text{wall}} = c_{g_r} |_{\perp \text{wall}} \]
\[ |\tilde{c}_{gi}| \sin(\tan^{-1} R + \tan^{-1} a) = |\tilde{c}_{gr}| \sin(\tan^{-1} R - \tan^{-1} a) \]

or
\[ \tilde{c}_{gr} = -\tilde{c}_{gi} \frac{(1 + aR)}{(1 - aR)} \]

While \( \lambda \) shortens in the reflection process, \( \tilde{c}_{gr} \) increases.

Notice that if \( aR > 1 \) the reflected \( \tilde{c}_{gr} \) is very large. What does this mean? It means that the bottom coincides with the outgoing characteristics: \( z = ax \rightarrow z = R^{-1}x \).

As \( aR \rightarrow 1 \), \( \tilde{c}_{gr} \) is very large, \( k_r \) is very large: the reflected wave is very small. The present inviscid analysis fails.

Rules for sloping bottom:

1. Angle \( \theta \) of \( \tilde{c}_{gi}, \tilde{c}_{gr} \) with the vertical must be the same.
2. The components \( \perp \) to bottom must be equal.