0.1 Ramsey problem

The production side is like in the Solow model. Output per capita

\[ y_t = f(k_t) \]

simplify \( n = 0 \) and \( g = 0 \) so the law of motion for capital per capita is

\[ k_{t+1} = (1 - \delta)k_t + i_t \]

\[ c_t + i_t = y_t \]

\( \implies k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t \)

\[ c_t = (1 - \delta)k_t + f(k_t) - k_{t+1} \tag{1} \]

with the constraints \( c_t \geq 0 \) and \( k_{t+1} \geq 0 \), and \( k_0 \) given.

In the Solow model, \( c_t = (1 - s)f(k_t) \). Now instead we consider the problem of how much the planner would consume/invest. The people in the economy derive utility from consuming, and so does the planner. For a consumption stream \( c = \{c_t\}_{t=0}^T \)

\[ \sum_{t=0}^T \beta^t u(c_t) \]

where \( \beta \in (0, 1) \) is the discount factor an captures impatience and we have an infinite horizon \( T = \infty \). \( u(c_t) \) is the per period utility function and we assume it is

- increasing
- concave
- Inada conditions \( \lim_{c \to 0} u'(c) = \infty \) and \( \lim_{c \to \infty} u'(c) = 0 \)
Example 1. For example, the CEIS function

\[ u(c) = \frac{c^{1-\frac{1}{\theta}}}{1-\frac{1}{\theta}} \]

where \( \theta > 0 \) is the EIS and controls how much the agent is willing to let his consumption vary across periods.

The Ramsey problem is

\[
\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t u \left( (1-\delta)k_t + f(k_t) \right) + \frac{c_t}{2} \\
\text{s.t.: } 0 \leq k_{t+1} \leq (1-\delta)k_t + f(k_t)
\]

\( k_0 \) given

Where if we have a solution \( \{k_t^*\} \) then we can rebuild the sequence of consumption \( \{c_t\} \) from (1) (and output \( \{y_t\} \) and investment \( \{i_t\} \)).

Finite Horizon \( T < \infty \). We can solve this problem in several ways. First imagine we have a finite horizon problem: \( t = 1, \ldots, T \).

Then we know how to solve this problem. Ignore the non-negativity constraints (we can check them later), taking FOC (because of the concavity, FOC will be sufficient) we get

\[
\beta^t u'(c_t)(-1) + \beta^{t+1} u'(c_{t+1}) ((1-\delta) + f'(k_{t+1})) = 0 \quad \forall t = 1 \ldots T - 1
\]

\[
u'(c_t) = \beta u'(c_{t+1}) ((1-\delta) + f'(k_{t+1}))
\]

The idea is that if we have a plan \( \{k_t\} \) and we decide to reduce consumption in period \( t \) by a small \( \epsilon \) and use that to invest and accumulate capital for next period \( k_{t+1} + \epsilon \) then we
can increase consumption next period by \((1 - \delta)\epsilon + f'(k_{t+1})\epsilon\) and keep \(k_{t+2}\) and the whole subsequent plan unchanged:

\[
\hat{c}_{t+1} = (1 - \delta)k_{t+1} + f(k_{t+1}) + (1 - \delta)\epsilon + f'(k_{t+1})\epsilon - k_{t+2}
\]

The reduction in consumption at time \(t\) has a cost in utility \(u'(c_t)\epsilon\) and from the increase in consumption in period \(t + 1\) we get \(\beta u'(c_{t+1})((1 - \delta) + f'(k_{t+1}))\epsilon\). It better be the case that we cannot improve by picking a small \(\epsilon\) (greater or smaller than zero), and this is what the FOC condition captures: “local deviations”

So we have a second order difference equation for \(\{k_t\}\):

\[
u'(\frac{c_t}{1 - \delta}k_t + f(k_t) - k_{t+1}) = \beta u'(\frac{c_{t+1}}{1 - \delta}k_{t+1} + f(k_{t+1}) - k_{t+2})((1 - \delta) + f'(k_{t+1}))
\]

with an initial condition \(k_0\) given. We need a second “boundary condition”. For the last period \(T\) we have \(k_{T+1} = 0\), so this has a unique solution \(\{k_t^*\}\). To understand this condition take the FOC for \(k_{T+1}\):

\[
\beta^T u'(c_T)(-1) \leq 0 \quad \text{and} \quad k_{T+1} \geq 0
\]

because the non-negativity constraint here can be binding (this is the Kuhn Tucker condition), and with complementary slackness:

\[
\beta^T u'(c_T)(-1)k_{T+1} = 0
\]

So if \(k_{T+1} > 0\), then \(\beta^{T+1}u'(c_T)(-1) = 0\) which cannot be. Hence, \(k_{T+1} = 0\). Intuitively, capital is worthless since the economy ends and we can’t use it to produce consumption goods.
If this condition failed, we could at some period $t < T$ consume a little more $c_t + \epsilon$ and obtain a little bit less capital next period but not make up for it with less consumption next period (keep the same consumption for every consecutive period), so that instead of keeping $k_{t+2}$ unchanged, it would go down a little, and so would all the consecutive $k_s$ for $s = t + 2...T$. If the original consecutive capital levels $k_s$ were strictly positive (and this will be the case for some $t$) this plan is still feasible, but better because we consumed more in one period and the same in all others! This is a “global” deviation.

Because of the concavity, the FOC - including the Kuhn Tucker inequality - are sufficient. Finally, after building the solution, check that the non-negativity constraints we ignored are actually satisfied and we are done.

**Infinite Horizon** $T = \infty$. With an infinite horizon $T = \infty$, we don’t have a “last period” and so we never want to have no capital. The second boundary condition becomes instead a “tranversality condition”:

$$\lim_{t \to \infty} \beta^t u'(c_t)k_{t+1} = 0$$

Proving this is beyond the scope of this course, but the intuition is similar to the finite horizon case: we don’t want to accumulate capital for its own sake. The FOC conditions make sure there are no “local” deviations (consuming a little bit less today and a little bit more tomorrow), the “tranversality condition” makes sure there is no “global” deviation, like simply consuming more today (without reducing consumption in the future) and having a little less capital in every consecutive period.
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