Lecture 9: Attitudes toward Risk

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Money Lotteries

Today: special case of choice under uncertainty where outcomes are measured in dollars.

Set of consequences $C$ is subset of $\mathbb{R}$.

A lottery is a cumulative distribution function $F$ on $\mathbb{R}$.

Assume preferences have expected utility representation:

$$U(F) = E_F [u(x)] = \int u(x) \, dF(x)$$

Assume $u$ increasing, differentiable.

**Question:** how do properties of von Neumann-Morgenstern utility function $u$ relate to decision-maker’s attitude toward risk?
Expected Value vs. Expected Utility

Expected value of lottery $F$ is

$$EF \ [x] = \int xdF \ (x)$$

Expected utility of lottery $F$ is

$$EF \ [u \ (x)] = \int u \ (x) \ dF \ (x)$$

Can learn about consumer’s risk attitude by comparing $EF \ [u \ (x)]$ and $u \ (EF \ [x])$. 
Risk Attitude: Definitions

Definition

A decision-maker is **risk-averse** if she always prefers the sure
wealth level \( E_F [x] \) to the lottery \( F \): that is,

\[
\int u(x) \, dF(x) \leq u \left( \int x \, dF(x) \right)
\]

for all \( F \).

A decision-maker is **strictly risk-averse** if the inequality is strict
for all non-degenerate lotteries \( F \).

A decision-maker is **risk-neutral** if she is always indifferent:

\[
\int u(x) \, dF(x) = u \left( \int x \, dF(x) \right)
\]

for all \( F \).

A decision-maker is **risk-loving** if she always prefers the lottery:

\[
\int u(x) \, dF(x) \geq u \left( \int x \, dF(x) \right)
\]

for all \( F \).
Risk Aversion and Concavity

Statement that \( \int u(x) \, dF(x) \leq u\left( \int x \, dF(x) \right) \) for all \( F \) is called \textbf{Jensen’s inequality}.

Fact: Jensen’s inequality holds iff \( u \) is concave.

This implies:

\textbf{Theorem}

A \textit{decision-maker} is (strictly) \textit{risk-averse} if and only if \( u \) is (strictly) concave.

A \textit{decision-maker} is \textit{risk-neutral} if and only if \( u \) is linear.

A \textit{decision-maker} is (strictly) \textit{risk-loving} if and only if \( u \) is (strictly) convex.
Certainty Equivalents

Can also define risk-aversion using certainty equivalents.

Definition
The certainty equivalent of a lottery $F$ is the sure wealth level that yields the same expected utility as $F$: that is,

$$CE(F, u) = u^{-1} \left( \int u(x) \, dF(x) \right).$$

Theorem
A decision-maker is risk-averse iff $CE(F, u) \leq E_F(x)$ for all $F$. A decision-maker is risk-neutral iff $CE(F, u) = E_F(x)$ for all $F$. A decision-maker is risk-loving iff $CE(F, u) \geq E_F(x)$ for all $F$. 
Quantifying Risk Attitude

We know what it means for a consumer to be risk-averse. What does it mean for one consumer to be more risk-averse than another?

Two possibilities:

1. \( u \) is more risk-averse than \( v \) if, for every \( F \),
   \[ CE(F, u) \leq CE(F, v) \).
2. \( u \) is more risk-averse than \( v \) if \( u \) is “more concave” than \( v \), in that \( u = g \circ v \) for some increasing, concave \( g \).

One more, based on local curvature of utility function:
\( u \) is more-risk averse than \( v \) if, for every \( x \),
\[
- \frac{u''(x)}{u'(x)} \geq - \frac{v''(x)}{v'(x)}
\]

\( A(x, u) = - \frac{u''(x)}{u'(x)} \) is called the Arrow-Pratt coefficient of absolute risk-aversion.
The following are equivalent:

1. For every $F$, $CE(F, u) \leq CE(F, v)$.

2. There exists an increasing, concave function $g$ such that $u = g \circ v$.

3. For every $x$, $A(x, u) \geq A(x, v)$.
How does risk attitude vary with wealth?

Natural to assume that a richer individual is more willing to bear risk: whenever a poorer individual is willing to accept a risky gamble, so is a richer individual.

Captured by decreasing absolute risk-aversion:

Definition
A von Neumann-Morenstern utility function $u$ exhibits decreasing (constant, increasing) absolute risk-aversion if $A(x, u)$ is decreasing (constant, increasing) in $x$. 
Theorem

Suppose $u$ exhibits decreasing absolute risk-aversion. If the decision-maker accepts some gamble at a lower wealth level, she also accepts it at any higher wealth level: that is, for any lottery $F(x)$, if

$$E_F[u(w + x)] \geq u(w),$$

then, for any $w' > w$,

$$E_F[u(w' + x)] \geq u(w').$$
Multiplicative Gambles

What about gambles that **multiply** wealth, like choosing how risky a stock portfolio to hold?

Are richer individuals also more willing to bear multiplicative risk?

Depends on increasing/decreasing **relative risk-aversion**:

\[
R(x, u) = - \frac{u''(x)}{u'(x)} x.
\]

**Theorem**

*Suppose* \( u \) *exhibits decreasing relative risk-aversion.*

*If the decision-maker accepts some multiplicative gamble at a lower wealth level, she also accepts it at any higher wealth level:*  
that is, for any lottery \( F(t) \), if

\[
E_F[u(tw)] \geq u(w),
\]

*then, for any* \( w' > w \),

\[
E_F[u(tw')] \geq u(w').
\]
Relative Risk-Aversion vs. Absolute Risk-Aversion

\[ R(x) = xA(x) \]

decreasing relative risk-aversion \(\implies\) decreasing absolute risk-aversion

increasing absolute risk-aversion \(\implies\) increasing relative risk-aversion

Ex. decreasing relative risk-aversion \(\implies\) more willing to gamble 1% of wealth as get richer.

So certainly more willing to gamble a fixed amount of money.
Application: Insurance

Risk-averse agent with wealth $w$, faces probability $p$ of incurring monetary loss $L$.

Can insure against the loss by buying a policy that pays out $a$ if the loss occurs.

Policy that pays out $a$ costs $qa$.

How much insurance should she buy?
Agent’s Problem

\[
\max_a pu(w - qa - L + a) + (1 - p) u(w - qa)
\]

\( u \) concave \( \implies \) concave problem, so FOC is necessary and sufficient.

FOC:

\[
p(1 - q) u'(w - qa - L + a) = (1 - p) qu'(w - qa)
\]

Equate marginal benefit of extra dollar in each state.
Actuarily Fair Prices

Insurance is **actuarily fair** if expected payout $qa$ equals cost of insurance $pa$: that is, $p = q$.

With acturarily fair insurance, FOC becomes

$$u'(w - qa - L + a) = u'(w - qa)$$

Solution: $a = L$

A risk-averse consumer facing actuarily fair prices will **always** fully insure.
Actuarily Unfair Prices

What if insurance company makes a profit, so \( q > p \)?

Rearrange FOC as

\[
\frac{u'(w - qa - L + a)}{u'(w - qa)} = \frac{(1 - p)q}{p(1 - q)} > 1
\]

Solution: \( a < L \)

A risk-averse consumer facing actuarily unfair prices will never fully insure.

Intuition: \( u \) approximately linear for small risks, so not worth giving up expected value to insure away last little bit of variance.
Comparative Statics

\[
\max_a pu (w - qa - L + a) + (1 - p) u (w - qa)
\]

Bigger loss $\implies$ buy more insurance ($a^*$ increasing in $L$)

Follows from Topkis’ theorem.

If agent has decreasing absolute risk-aversion, then she buys less insurance as she gets richer.

See notes for proof.
Application: Portfolio Choice

Risk-averse agent with wealth $w$ has to invest in a safe asset and a risky asset.

Safe asset pays certain return $r$.

Risky asset pays random return $z$, with cdf $F$.

Agent’s problem

$$\max_{a \in [0,w]} \int u\left(az + (w - a)r\right) dF(z)$$

First-order condition

$$\int (z - r) u'\left(az + (w - a)r\right) dF(z) = 0$$
Risk-Neutral Benchmark

Suppose \( u' (x) = \alpha x \) for some \( \alpha > 0 \).

Then
\[
U (a) = \int \alpha (az + (w - a)r) \, dF (z),
\]
so
\[
U' (a) = \alpha (E[z] - r).
\]

Solution: set \( a = w \) if \( E[z] > r \), set \( a = 0 \) if \( E[z] < r \).

Risk-neutral investor puts all wealth in the asset with the highest rate of return.
If safe asset has higher rate of return, then even risk-averse investor puts all wealth in the safe asset.
More Interesting Case

What if agent is risk-averse, but risky asset has higher expected return?

\[ U'(0) = (E[z] - r) u'(w) > 0 \]

If risky asset has higher rate of return, then risk-averse investor always puts some wealth in the risky asset.
Comparative Statics

Does a less risk-averse agent always invest more in the risky asset?

Sufficient condition for agent $v$ to invest more than agent $u$:

$$\int (z - r) u'(az + (w - a)r) \, dF = 0$$

$$\implies \int (z - r) v'(az + (w - a)r) \, dF \geq 0$$

$u$ more risk-averse $\implies v = h \circ u$ for some increasing, convex $h$.

Inequality equals

$$\int (z - r) h'(u(az + (w - a)r)) u'(az + (w - a)r) \, dF \geq 0$$

$h'(\cdot)$ positive and increasing in $z$

$\implies$ multiplying by $h'(\cdot)$ puts more weight on positive $(z > r)$ terms, less weight on negative terms.

A less risk-averse agent always invests more in the risky asset.
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