Single-Deviation Principle and Bargaining

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Multi-stage games with observable actions

- finite set of players $N$
- stages $t = 0, 1, 2, \ldots$
- $H$: set of terminal histories (sequences of action profiles of possibly different lengths)
- at stage $t$, after having observed a non-terminal history of play $h_t = (a^0, \ldots, a^{t-1}) \not\in H$, each player $i$ simultaneously chooses an action $a^t_i \in A_i(h_t)$
- $u_i(h)$: payoff of $i \in N$ for terminal history $h \in H$
- $\sigma_i$: behavior strategy for $i \in N$ specifies $\sigma_i(h) \in \Delta(A_i(h))$ for $h \not\in H$

Often natural to identify “stages” with time periods.

Examples

- repeated games
- alternating bargaining game
Unimprovable Strategies

To verify that a strategy profile \( \sigma \) constitutes a subgame perfect equilibrium (SPE) in a multi-stage game with observed actions, it suffices to check whether there are any histories \( h_t \) where some player \( i \) can gain by deviating from playing \( \sigma_i(h_t) \) at \( t \) and conforming to \( \sigma_i \) elsewhere.

\( u_i(\sigma|h_t) \): expected payoff of player \( i \) in the subgame starting at \( h_t \) and played according to \( \sigma \) thereafter

**Definition 1**

A strategy \( \sigma_i \) is *unimprovable* given \( \sigma_{-i} \) if \( u_i(\sigma_i, \sigma_{-i}|h_t) \geq u_i(\sigma'_i, \sigma_{-i}|h_t) \) for every \( h_t \) and \( \sigma'_i \) with \( \sigma'_i(h) = \sigma_i(h) \) for all \( h \neq h_t \).
Continuity at Infinity

If $\sigma$ is an SPE then $\sigma_i$ is unimprovable given $\sigma_{-i}$. For the converse...

**Definition 2**

A game is **continuous at infinity** if

$$\lim_{t \to \infty} \sup_{(h, \tilde{h}) \mid h_t = \tilde{h}_t} \left| u_i(h) - u_i(\tilde{h}) \right| = 0, \forall i \in \mathbb{N}.$$ 

Events in the distant future are relatively unimportant.
Theorem 1

Consider a multi-stage game with observed actions that is continuous at infinity. If \( \sigma_i \) is unimprovable given \( \sigma_{-i} \) for all \( i \in N \), then \( \sigma \) constitutes an SPE.

Proof allows for infinite action spaces at some stages. There exist versions for games with unobserved actions.
Proof

Suppose that $\sigma_i$ is unimprovable given $\sigma_{-i}$, but $\sigma_i$ is not a best response to $\sigma_{-i}$ following some history $h_t$. Let $\sigma_i^1$ be a strictly better response and define

$$\varepsilon = u_i(\sigma_i^1, \sigma_{-i}|h_t) - u_i(\sigma_i, \sigma_{-i}|h_t) > 0.$$  

Since the game is continuous at infinity, there exists $t' > t$ and $\sigma_i^2$ s.t. $\sigma_i^2$ is identical to $\sigma_i^1$ at all information sets up to (and including) stage $t'$, $\sigma_i^2$ coincides with $\sigma_i$ across all longer histories and

$$|u_i(\sigma_i^2, \sigma_{-i}|h_t) - u_i(\sigma_i^1, \sigma_{-i}|h_t)| < \varepsilon/2.$$  

Then

$$u_i(\sigma_i^2, \sigma_{-i}|h_t) > u_i(\sigma_i, \sigma_{-i}|h_t).$$
Proof

$\sigma^3_i$: strategy obtained from $\sigma^2_i$ by replacing the stage $t'$ actions following any history $h_{t'}$ with the corresponding actions under $\sigma_i$

Conditional on any $h_{t'}$, $\sigma_i$ and $\sigma^3_i$ coincide, hence

$$u_i(\sigma^3_i, \sigma_{-i}|h_{t'}) = u_i(\sigma_i, \sigma_{-i}|h_{t'}).$$

As $\sigma_i$ is unimprovable given $\sigma_{-i}$, and conditional on $h_{t'}$ the subsequent play in strategies $\sigma_i$ and $\sigma^2_i$ differs only at stage $t'$,

$$u_i(\sigma_i, \sigma_{-i}|h_{t'}) \geq u_i(\sigma^2_i, \sigma_{-i}|h_{t'}).$$

Then

$$u_i(\sigma^3_i, \sigma_{-i}|h_{t'}) \geq u_i(\sigma^2_i, \sigma_{-i}|h_{t'})$$

for all histories $h_{t'}$. Since $\sigma^2_i$ and $\sigma^3_i$ coincide before reaching stage $t'$,

$$u_i(\sigma^3_i, \sigma_{-i}|h_t) \geq u_i(\sigma^2_i, \sigma_{-i}|h_t).$$
Proof

\(\sigma^4_i\): strategy obtained from \(\sigma^3_i\) by replacing the stage \(t' - 1\) actions following any history \(h_{t'-1}\) with the corresponding actions under \(\sigma_i\).

Similarly,

\[ u_i(\sigma^4_i, \sigma_{-i}|h_t) \geq u_i(\sigma^3_i, \sigma_{-i}|h_t) \ldots \]

The final strategy \(\sigma^{t'-t+3}_i\) is identical to \(\sigma_i\) conditional on \(h_t\) and

\[ u_i(\sigma_i, \sigma_{-i}|h_t) = u_i(\sigma^{t'-t+3}_i, \sigma_{-i}|h_t) \geq \ldots \]

\[ \geq u_i(\sigma^3_i, \sigma_{-i}|h_t) \geq u_i(\sigma^2_i, \sigma_{-i}|h_t) > u_i(\sigma_i, \sigma_{-i}|h_t), \]

a contradiction.
Apply the single deviation principle to repeated prisoners’ dilemma to implement the following equilibrium paths for high discount factors:

- \((C, C), (C, C), \ldots\)
- \((C, C), (C, C), (D, D), (C, C), (C, C), (D, D), \ldots\)
- \((C, D), (D, C), (C, D), (D, C) \ldots\)

Cooperation is possible in repeated play.
Rubinstein (1982)

- players \( i = 1, 2; j = 3 - i \)
- set of feasible utility pairs

\[
U = \{(u_1, u_2) \in [0, \infty)^2 | u_2 \leq g_2(u_1)\}
\]

- \( g_2 \) s. decreasing, concave (and hence continuous), \( g_2(0) > 0 \)
- \( \delta_i \): discount factor of player \( i \)
- at every time \( t = 0, 1, \ldots \), player \( i(t) \) proposes an alternative \( u = (u_1, u_2) \in U \) to player \( j(t) = 3 - i(t) \)

\[
i(t) = \begin{cases} 
1 & \text{for } t \text{ even} \\
2 & \text{for } t \text{ odd}
\end{cases}
\]

- if \( j(t) \) accepts the offer, game ends yielding payoffs \( (\delta^t_1 u_1, \delta^t_2 u_2) \)
- otherwise, game proceeds to period \( t + 1 \)
Stationary SPE

Define \( g_1 = g_2^{-1} \). Graphs of \( g_2 \) and \( g_1^{-1} \): Pareto-frontier of \( U \)

Let \((m_1, m_2)\) be the unique solution to the following system of equations

\[
\begin{align*}
  m_1 &= \delta_1 g_1 (m_2) \\
  m_2 &= \delta_2 g_2 (m_1).
\end{align*}
\]

\((m_1, m_2)\) is the intersection of the graphs of \( \delta_2 g_2 \) and \((\delta_1 g_1)^{-1}\).

SPE in “stationary” strategies: in any period where player \( i \) has to make an offer to \( j \), he offers \( u \) with \( u_j = m_j \) and \( u_i = g_i(m_j) \), and \( j \) accepts only offers \( u \) with \( u_j \geq m_j \).

*Single-deviation principle*: constructed strategies form an SPE.

Is the SPE unique?
Iterated Conditional Dominance

Definition 3
In a multi-stage game with observable actions, an action $a_i$ is conditionally dominated at stage $t$ given history $h_t$ if, in the subgame starting at $h_t$, every strategy for player $i$ that assigns positive probability to $a_i$ is strictly dominated.

Proposition 1
In any multi-stage game with observable actions, every SPE survives the iterated elimination of conditionally dominated strategies.
Equilibrium uniqueness

*Iterated conditional dominance:* stationary equilibrium is essentially the unique SPE.

**Theorem 2**

*The SPE of the alternating-offer bargaining game is unique, except for the decision to accept or reject Pareto-inefficient offers.*
Proof

▶ Following a disagreement at date $t$, player $i$ cannot obtain a period $t$ expected payoff greater than

$$M_i^0 = \delta_i \max_{u \in U} u_i = \delta_i g_i(0)$$

▶ Rejecting an offer $u$ with $u_i > M_i^0$ is conditionally dominated by accepting such an offer for $i$.

▶ Once we eliminate dominated actions, $i$ accepts all offers $u$ with $u_i > M_i^0$ from $j$.

▶ Making any offer $u$ with $u_i > M_i^0$ is dominated for $j$ by an offer

$$\bar{u} = \lambda u + (1 - \lambda) \left( M_i^0, g_j \left( M_i^0 \right) \right)$$

for $\lambda \in (0, 1)$ (both offers are accepted immediately).
Proof

Under the surviving strategies

▶ $j$ can reject an offer from $i$ and make a counteroffer next period that leaves him with slightly less than $g_j(M^0_i)$, which $i$ accepts; it is conditionally dominated for $j$ to accept any offer smaller than

$$m^1_j = \delta_j g_j(M^0_i)$$

▶ $i$ cannot expect to receive a continuation payoff greater than

$$M^1_i = \max(\delta_i g_i(m^1_j), \delta_i^2 M^0_i) = \delta_i g_i(m^1_j)$$

after rejecting an offer from $j$

$$\delta_i g_i(m^1_j) = \delta_i g_i(\delta_j g_j(M^0_i)) \geq \delta_i g_i(g_j(M^0_i)) = \delta_i M^0_i \geq \delta_i^2 M^0_i$$
Proof

Recursively define

\[ m_j^{k+1} = \delta_j g_j (M_i^k) \]
\[ M_i^{k+1} = \delta_i g_i (m_j^{k+1}) \]

for \( i = 1, 2 \) and \( k \geq 1 \). \( (m_i^k)_{k \geq 0} \) is increasing and \( (M_i^k)_{k \geq 0} \) is decreasing.

Prove by induction on \( k \) that, under any strategy that survives iterated conditional dominance, player \( i = 1, 2 \)

- never accepts offers with \( u_i < m_i^k \)
- always accepts offers with \( u_i > M_i^k \), but making such offers is dominated for \( j \).
Proof

- The sequences \((m_i^k)\) and \((M_i^k)\) are monotonic and bounded, so they need to converge. The limits satisfy

\[
\begin{align*}
m_j^\infty &= \delta_j g_j \left( \delta_i g_i \left( m_j^\infty \right) \right) \\
M_i^\infty &= \delta_i g_i \left( m_j^\infty \right).
\end{align*}
\]

- \((m_1^\infty, m_2^\infty)\) is the (unique) intersection point of the graphs of the functions \(\delta_2 g_2\) and \((\delta_1 g_1)^{-1}\)

- \(M_i^\infty = \delta_i g_i \left( m_j^\infty \right) = m_i^\infty\)

- All strategies of \(i\) that survive iterated conditional dominance accept \(u\) with \(u_i > M_i^\infty = m_i^\infty\) and reject \(u\) with \(u_i < m_i^\infty = M_i^\infty\).
Proof

In an SPE

- at any history where \( i \) is the proposer, \( i \)'s payoff is at least \( g_i(m_j^\infty) \): offer \( u \) arbitrarily close to \( (g_i(m_j^\infty), m_j^\infty) \), which \( j \) accepts under the strategies surviving the elimination process
- \( i \) cannot get more than \( g_i(m_j^\infty) \)
  - any offer made by \( i \) specifying a payoff greater than \( g_i(m_j^\infty) \) for himself would leave \( j \) with less than \( m_j^\infty \); such offers are rejected by \( j \) under the surviving strategies
  - under the surviving strategies, \( j \) never offers \( i \) more than \( M_i^\infty = \delta_i g_i(m_j^\infty) \leq g_i(m_j^\infty) \)
  - hence \( i \)'s payoff at any history where \( i \) is the proposer is exactly \( g_i(m_j^\infty) \); possible only if \( i \) offers \( (g_i(m_j^\infty), m_j^\infty) \) and \( j \) accepts with probability 1

Uniquely pinned down actions at every history, except those where \( j \) has just received an offer \( (u_i, m_j^\infty) \) for some \( u_i < g_i(m_j^\infty) \)
Properties of the equilibrium

- The SPE is efficient—agreement is obtained in the first period, without delay.
- SPE payoffs: \((g_1(m_2), m_2)\), where \((m_1, m_2)\) solve

\[
\begin{align*}
m_1 &= \delta_1 g_1 (m_2) \\
m_2 &= \delta_2 g_2 (m_1)
\end{align*}
\]

- Patient players get higher payoffs: the payoff of player \(i\) is increasing in \(\delta_i\) and decreasing in \(\delta_j\).
- For a fixed \(\delta_1 \in (0, 1)\), the payoff of player 2 converges to 0 as \(\delta_2 \to 0\) and to \(\max_{u \in U} u_2\) as \(\delta_2 \to 1\).
- If \(U\) is symmetric and \(\delta_1 = \delta_2\), player 1 enjoys a first mover advantage: \(m_1 = m_2\) and \(g_1(m_2) = m_2/\delta > m_2\).
Nash Bargaining

Assume $g_2$ is decreasing, s. concave and continuously differentiable.

_Nash (1950) bargaining solution:_

$$\{u^*\} = \arg \max_{u \in U} u_1 u_2 = \arg \max_{u \in U} u_1 g_2(u_1).$$

**Theorem 3 (Binmore, Rubinstein and Wolinsky 1985)**

Suppose that $\delta_1 = \delta_2 =: \delta$ in the alternating bargaining model. Then the unique SPE payoffs converge to the Nash bargaining solution as $\delta \to 1$.

$$m_1 g_2 (m_1) = m_2 g_1 (m_2)$$

$(m_1, g_2 (m_1))$ and $(g_1 (m_2), m_2)$ belong to the intersection of $g_2$’s graph with the same hyperbola, which approaches the hyperbola tangent to the boundary of $U$ (at $u^*$) as $\delta \to 1$. 
Bargaining with random selection of proposer

- Two players need to divide $1.
- Every period $t = 0, 1, \ldots$ player 1 is chosen with probability $p$ to make an offer to player 2.
- Player 2 accepts or rejects 1’s proposal.
- Roles are interchanged with probability $1 - p$.
- In case of disagreement the game proceeds to the next period.
- The game ends as soon as an offer is accepted.
- Player $i = 1, 2$ has discount factor $\delta_i$. 
The unique equilibrium is stationary, i.e., each player \( i \) has the same expected payoff \( v_i \) in every subgame.

Payoffs solve

\[
\begin{align*}
v_1 &= p(1 - \delta_2 v_2) + (1 - p)\delta_1 v_1 \\
v_2 &= p\delta_2 v_2 + (1 - p)(1 - \delta_1 v_1).
\end{align*}
\]

The solution is

\[
\begin{align*}
v_1 &= \frac{p/(1 - \delta_1)}{p/(1 - \delta_1) + (1 - p)/(1 - \delta_2)} \\
v_2 &= \frac{(1 - p)/(1 - \delta_2)}{p/(1 - \delta_1) + (1 - p)/(1 - \delta_2)}.
\end{align*}
\]
Comparative Statics

\[
\begin{align*}
\nu_1 & = \frac{1}{1 + \frac{(1-p)(1-\delta_1)}{p(1-\delta_2)}} \\
\nu_2 & = \frac{1}{1 + \frac{p(1-\delta_2)}{(1-p)(1-\delta_1)}}.
\end{align*}
\]

- Immediate agreement
- First mover advantage
  - \( \nu_1 \) increases with \( p \), \( \nu_2 \) decreases with \( p \).
  - For \( \delta_1 = \delta_2 \), we obtain \( \nu_1 = p, \nu_2 = 1 - p \).
- Patience pays off
  - \( \nu_i \) increases with \( \delta_i \) and decreases with \( \delta_j \) \((j = 3 - i)\).
  - Fix \( \delta_j \) and take \( \delta_i \rightarrow 1 \), we get \( \nu_i \rightarrow 1 \) and \( \nu_j \rightarrow 0 \).
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