1.5 Long run contracts with limited enforcement

• See Hopenhayn and Clementi (2006) for something close to the moral-hazard model of Holmstrom and Tirole

• Here follow Lorenzoni and Walentin (2007)

• Process $A_t = \Gamma (A_{t-1}, \epsilon_t)$

• Consumers and entrepreneurs, unit mass of each

• Consumers have endowment of labor $l_C$

• Risk neutral, discount factor $\beta_E < \beta_C$
• Entrepreneurs die and replaced with prob $\gamma$

• Newborn entrepreneur supply $l_E$, start with $n = wl_E$

• Offer long-term financial contract $\{d_t\}_{t=t_0}^\infty$

• Market for used capital $q^0_t$

• Total adjustment cost $G(k_{t+1}, k^0_t)$
Budget constraint:

- **first period of life**
  \[ c_t^E + G(k_{t+1}, k_t^o) + q_t^o k_t^o \leq w_t l_E - d_t \]

- **continuation period**
  \[ c_t^E + G(k_{t+1}, k_t^o) + q_t^o (k_t^o - k_t) \leq A_t F(k_t, l_t) - w_t l_t - d_t \]

- **last period**
  \[ c_t^E = A_t F(k_t, l_t) - w_t l_t + q_t^o k_t - d_t. \]
Same trick as above (CRS)

- first period of life

\[ c_t^E + q_t^m k_{t+1}^o + q_t^o k_t^o \leq w_t l_E - d_t \]

- continuation period

\[ c_t^E + q_t^m k_{t+1}^o \leq [A_t F (k_t, l_t) - w_t l_t + q_t^o k_t] - d_t \]
\[ = R_t k_t - d_t \]

- last period

\[ c_t^E = R_t k_t - d_t. \]
1.5.1 Limited enforcement

- Entrepreneur controls firm’s assets

- In each period, can run away, diverting a fraction \((1 - \theta)\)

- If he does so, he re-enters the financial market as a young entrepreneur, with initial wealth

\[(1 - \theta) R_t k_t\]

and no debt
1.5.2 Recursive competitive equilibrium

- Aggregate state variables

\[ X_t \equiv (A_t, K_t, B_t) \]

- Conjecture: positive consumers’ consumption

- Present value of the liabilities of individual entrepreneur

\[ b_t = \mathbb{E}_t \left[ \sum_{s=0}^{\infty} \beta^s C^d_{t+s} \right] \]

- \( B_t \) economy-wide aggregate of these liabilities
• Recursive CE: law of motions for the endogenous state variables

\[ K_t = \mathcal{K}(X_{t-1}), \]
\[ B_t = \mathcal{B}(X_{t-1}, \epsilon_t), \]

and maps

\[ w(X_t), q^o(X_t) \]

• Use compact notation

\[ X_t = H(X_{t-1}, \epsilon_t). \]
1.5.3 Optimal financial contracts

- Continuing entrepreneur, in state $X$, who controls a firm with capital $k$ and outstanding liabilities $b$

- $V(k, b, X)$ expected utility computed
  - after production takes place
  - assuming the entrepreneur has chosen no default
  - before new investment and consumption
• Budget constraint

\[ c^E + q^m (X) k' \leq R (X) k - d \]

• Promise-keeping constraint

\[ b = d + \beta_C \left( (1 - \gamma) \sum \pi (\epsilon') b' (\epsilon') + \gamma \sum \pi (\epsilon') b'_L (\epsilon') \right) \]

• \( b' (\epsilon') , b'_L (\epsilon') \) PV of liabilities next period
• No-default condition

\[ V(k', b'(\epsilon'), X') \geq V((1 - \theta)k', 0, X') \]

for all \( \epsilon' \) and \( X' = H(X, \epsilon') \).

• If tomorrow final period, no-default

\[ R(X')k' - b'_{L}(\epsilon') \geq (1 - \theta)R(X')k' \]
Conjecture: linear value function

\[ V(k, b, X) = \phi(X)(R(X)k - b) \]

linear in net worth:

\[ R(X)k - b \]

- Then no default becomes

\[
\begin{align*}
  b'(\epsilon') & \leq \theta R(H(X, \epsilon'))k' \\
  b'_L(\epsilon') & \leq \theta R(H(X, \epsilon'))k'
\end{align*}
\]

for all \( \epsilon' \)
Bellman equation

\[ V(k, b, X) = \max_{c^E, k', b'(\cdot), b'_L(\cdot)} c^E + \beta_E (1 - \gamma) \sum \pi(\epsilon') V(k', b'(\epsilon'), H(X, \epsilon')) + \beta_E \gamma \sum \pi(\epsilon') [R(H(X, \epsilon')) k' - b'_L(\epsilon')] \]

\[ c^E + q^m(X) k' \leq R(X) k - d \]

\[ b = d + \beta_C \left( (1 - \gamma) \sum \pi(\epsilon') b'(\epsilon') + \gamma \sum \pi(\epsilon') b'_L(\epsilon') \right) \]

\[ b'(\epsilon') \leq \theta R(H(X, \epsilon')) k' \]

\[ b'_L(\epsilon') \leq \theta R(H(X, \epsilon')) k' \]
Assumptions

• profitability

$$\beta_E \mathbb{E} \left[ R \left( H \left( X, \epsilon' \right) \right) \right] > q^m (X)$$  \hspace{1cm} (a)

• limited pledgeability

$$\theta \beta_C \mathbb{E} \left[ R \left( H \left( X, \epsilon' \right) \right) \right] < q^m (X)$$  \hspace{1cm} (b)

• finite utility

$$\frac{(1 - \gamma)(1 - \theta) \mathbb{E} \left[ R \left( H \left( X, \epsilon' \right) \right) \right]}{q^m (X) - \theta \beta_C \mathbb{E} \left[ R \left( H \left( X, \epsilon' \right) \right) \right]} < 1$$  \hspace{1cm} (c)
Then find
\[
\phi (X) = \frac{\beta_E (1 - \theta) \mathbb{E} [(\gamma + (1 - \gamma) \phi (H (X, \epsilon')))] R (H (X, \epsilon'))}{q^m (X) - \theta \beta_C \mathbb{E} [R (H (X, \epsilon'))]}
\]
and guess and verify that:
\[
V (k, b, X) = \phi (X) (R (X) k - b)
\]
Optimal solution

- no consumption until final date
  \[ c^E = 0, \]

- maximum borrowing
  \[ b' (\epsilon') = b'_L (\epsilon') = \theta R (H (X, \epsilon')) k'. \]

- dynamics for capital accumulation
  \[ k' = \frac{R (X) k - b}{q^m (X) - \theta \beta E [R (H (X, \epsilon'))]} \]
Need one extra assumption (no delay):

$$\phi(X) > \frac{\beta_E}{\beta_C} \phi(H(X, \epsilon'))$$

(d)
1.5.4 Aggregation

\[ N_t = (1 - \gamma)(R_tK_t - B_t) + \gamma w_t l_E \]

\[ K_{t+1} = \frac{(1 - \gamma)(R_tK_t - B_t) + \gamma w_t l_E}{q_t^m - \theta \beta C E_t [R_{t+1}]} \]

\[ B_{t+1} = \beta C \theta R_{t+1} K_{t+1} \]

\[ w_t = A_t \frac{\partial F(K_t, 1)}{\partial L_t} \]

\[ q_t^o = -\frac{\partial G(K_{t+1}, K_t)}{\partial K_t} \]
• This confirms that the state variables in $X$ are sufficient to characterize the dynamics of prices

• Aggregation relies on linearity

• Trades of used capital:
  
  – entering entrepreneurs buy
    
    \[
    \frac{\gamma w_t l_E}{q_t^m - \theta \beta C E_t [R_{t+1}] k_{t+1}} \quad k_t^0
    \]

  – exiting entrepreneurs sell
    
    \[(1 - \gamma) K_t\]
Computation

2nd order stoch. difference equation in $K_t$

$$K_{t+1} = \frac{(1 - \gamma)(1 - \theta) R_t K_t + \gamma w_t l_E}{q_t^m - \theta \beta C^E_t [R_{t+1}]}$$

with

$$w_t = A_t \frac{\partial F (K_t, 1)}{\partial L_t}$$

$$R_t = A_t \frac{\partial F (K_t, 1)}{\partial K_t} - \frac{\partial G (K_{t+1}, K_t)}{\partial K_t}$$

$$q_t^m = \frac{\partial G (K_{t+1}, K_t)}{\partial K_{t+1}}$$

• Remember to check that (a)-(d) are satisfied!
\[ A_t F (k_t, l_t) = A_t k_t^\alpha l_t^{1-\alpha}, \]
\[ G (k_{t+1}, k_t) = k_{t+1} - (1 - \delta) k_t + \frac{\xi (k_{t+1} - k_t)^2}{2 k_t}. \]
1.5.5 Steady state

\[ A_t = 1 \]

\[(1 - \theta \beta_C R^S) K^S = (1 - \gamma)(1 - \theta) R^S K^S + \gamma w^S l_E \]

\[ R^S = \alpha (K^S)^{\alpha-1} + 1 - \delta \]

\[ K^S = \left( \frac{\alpha (\theta \beta_C + (1 - \gamma)(1 - \theta)) + \gamma (1 - \alpha) l_E}{1 - (\theta \beta_C + (1 - \gamma)(1 - \theta))(1 - \delta)} \right)^{\frac{1}{1-\alpha}} \]

Parameters such that (a)

\[ \beta_E R^S > 1 \]
then (b) and (c)

\[ \theta \beta_C R^S < 1 \text{ and } \frac{(1 - \gamma)(1 - \theta)}{1 - \theta \beta_C R^S} < 1 \]

follow from \( \gamma w^S l_E > 0 \)

\[ 1 - \beta_C \theta R^S - (1 - \gamma)(1 - \theta) R^S > 0. \]

In steady state \( \phi(X) \) is

\[ \phi^S = \frac{(1 - \theta) \beta_E R^S}{1 - \theta \beta_C R^S} \left( \gamma + (1 - \gamma) \phi^{S'} \right) \]

Condition (d) is why \( \beta_E < \beta_C \) is needed.
1.5.6 Frictionless benchmark

Very close to Hayashi (1982)

- entrepreneurs consume $w_t l_E$ in first period of life
- all investment financed with outside funds
- capital stock dynamics

$$\beta C E_t [R_t] = q_t^m$$

$$q_t^m = q_t$$
1.5.7 Q theory

• value of the firm (end of period)

\[ p_t = V (k_t, b_t, X_t) + b_t - c_t^E - d_t = \]
\[ = \phi_t (R_t k_t - b_t) + b_t - d_t \]
\[ = (\phi_t - 1) (R_t k_t - b_t) + q_t^m k_{t+1} \]

• Tobin’s q

\[ q_t = (\phi_t - 1) \frac{R_t k_t - b_t}{k_{t+1}} + q_t^m > q_t^m \]

Recall that \( \phi_t \) is forward looking variable capturing future excess returns

\[ \phi_t = \frac{\beta E (1 - \theta) \mathbb{E}_t \left[ (\gamma + (1 - \gamma) \phi_{t+1}) R_{t+1} \right]}{q_t^m - \theta \beta C \mathbb{E}_t [R_{t+1}]} \]
<table>
<thead>
<tr>
<th>Model</th>
<th>$a_1$</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model with financial friction</td>
<td>0.018</td>
<td>0.444</td>
</tr>
<tr>
<td>Frictionless model</td>
<td>0.118</td>
<td>0.000</td>
</tr>
<tr>
<td>Gilchrist and Himmelberg (1995)</td>
<td>0.033 (0.016)</td>
<td>0.242 (0.038)</td>
</tr>
</tbody>
</table>
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1.5.8 Wrapping up on q theory

- financial frictions can help explain failure of q-theory equations

- disconnect between when funds available and when profitable investment opportunities arise

- related ideas: growth options (Abel and Eberly)