Notes on Tax Implementation

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1 Certainty

• utility

\[ U(x, \theta) \]

where \( x \in X \) is a vector and \( \theta \in \Theta \) is worker types

• Example 1: Mirrlees (1971) has \( x = (c, -y) \) where \( c \) is consumption \( y \) is effective labor; in this case we want to know study the non-linear income tax schedule.

• Example 2: Two-period model with labor in the first period and consumption in both periods: \( U(c_0, c_1, y_0) \); in this example we’d like to study the nonlinear taxation of income and the taxation of savings [Atkinson-Stiglitz applies if \( U \) is separable]

• at this stage: no assumption on preferences (conavity, dimentionality of \( \Theta \), single crossing, etc.) needed

• define MRS

\[ MRS_{ij}(x, \theta) = \frac{U_i(x, \theta)}{U_j(x, \theta)} \]

• when \( \hat{x}(\theta) \) is an optimal allocation, it to look at the “wedges” or “implicit marginal taxes” defined by either

\[ \frac{MRS_{ij}(\hat{x}(\theta), \theta)}{p_j} \quad \text{or} \quad \frac{MRS_{ij}(\hat{x}(\theta), \theta)}{p_j/p_i} \]

we want to understand to what extent these measures are related to explicit taxes
1.1 The Problem of Implementation...

- incentive compatible allocation is a function $\hat{x} : \Theta \to X$ such that

$$U(\hat{x}(\theta), \theta) \geq U(\hat{x}(\theta'), \theta) \quad \forall \theta, \theta' \in \Theta^2$$

(1)

- implementability question: what budget sets $B$ can we confront agent with and get $\hat{x}$ allocation?

$$\hat{x}(\theta) \in \arg \max_{x \in B} U(x, \theta)$$

(2)

- Note: $B$ is independent of $\theta$

...captures anonymous taxation

1.2 ...Its Solution...

- smallest set that works...

$$\underline{B} \equiv \{ x \mid \exists \theta \in \Theta \text{ s.t. } x = \hat{x}(\theta) \}$$


- this gives as much choice as the direct mechanism!...

... not a lot of choice if $X$ has high dimension and $\Theta$ is low dimension

- largest set?

$$\bar{B} \equiv \{ x \mid U(x, \theta) \leq U(\hat{x}(\theta), \theta) \forall \theta \in \Theta \}$$

equivalently

$$\bar{B} \equiv \{ x \mid U(x, \theta) \leq \hat{v}(\theta) \forall \theta \in \Theta \}$$

where $\hat{v}(\theta) \equiv U(\hat{x}(\theta), \theta)$

- full characterization: any set $B$ such that

$$B \subseteq B \subseteq \bar{B}$$

also implements $\hat{x}$

1.3 ...In Terms of Taxes

- to think of taxation...
- benchmark budget without tax:

\[ p \cdot x \leq 0 \]

- \( T(x) \) function such that

\[ p \cdot x + T(x) \leq 0 \]

is equivalent to \( x \in B \) where \( B \) implements \( \hat{x} \)

- for lowest possible taxes use \( B = \bar{B} \)

\begin{itemize}
  \item numeraire good: \( x = (x_1, x_{-1}) \) with \( p_1 = 1 \) then

\[ x_1 + T(x_{-1}) + p_{-1} \cdot x_{-1} \leq 0 \]

\end{itemize}

- retention function...

\[ x_1 \leq R(x_{-1}) = -(T(x_{-1}) + p_{-1} \cdot x_{-1}) \]

- to implement we need

\[ \hat{x}_1(\theta) = R(\hat{x}_{-1}(\theta)) \quad \theta \in \Theta \]

and

\[ R(x_{-1}) \leq \hat{R}(x_{-1}) \equiv \max_{x_1} x_1 \quad \text{s.t.} \quad U(x_1, x_{-1}, \theta) \leq \hat{v}(\theta) \quad \forall \theta \in \Theta \]

- equivalently: need \( R(x_{-1}) \leq \hat{R}(x_{-1}) \) for all \( x \in X \) and \( R(x_{-1}) = \hat{R}(x_{-1}) \) for \( x \in B \).

- invert...

\[ U(x_1, x_{-1}, \theta) \leq \hat{v}(\theta) \]

to write...

\[ x_1 \leq U^{-1}(\hat{v}(\theta), x_{-1}, \theta) \]

- then

\[ \hat{R}(x_{-1}) \equiv \min_{\theta \in \Theta} U^{-1}(\hat{v}(\theta), x_{-1}, \theta) \]

### 1.4 Some Properties of the Solution

- idea: since \( \hat{R} \) defined as optimization, we can apply Maximum and Envelope Theorems
• economic questions...
  – how much more choice?
  – marginal taxes exist?
  – do they equal wedges?

• **Maximum Theorem:** Assume \( U : X \times \Theta \rightarrow \mathbb{R} \) is continuous, then \( \hat{\theta}(\theta) \) and \( \hat{R}(x_{-1}) \) are continuous functions; the set

\[
M(x_{-1}) \equiv \arg\min_{\theta \in \Theta} U^{-1}(\hat{\theta}(\theta), x_{-1}, \theta)
\]

is upper hemi continuous correspondence (note that \( \theta \in M(\hat{x}_{-1}(\theta)) \))

• this means we never impose sharp penalties in the sense of discontinuous taxes;

• In contrast, the direct mechanism implicitly imposes infinite taxes for any allocation outside \( B \)! In this sense, Taxes are very discontinous.

• **Envelope Theorem:** Suppose \( U \) is differentiable w.r.t. \( x \) and \( M(x_{-1}) \) is single valued, then

\[
\frac{\partial}{\partial x_{-1}} \hat{R}(x_{-1}) \equiv \frac{\partial}{\partial x_{-1}} U^{-1}(\hat{\theta}(M(x_{-1})), x_{-1}, M(x_{-1})) = -\frac{\partial}{\partial x_{-1}} U(\hat{R}(x_{-1}), x_{-1}, M(x_{-1}))
\]

That is,

\[
\frac{\partial}{\partial x_{-1}} \hat{R}(x_{-1}) = MRS_{x_1, x_{-1}}(\hat{R}(x_{-1}), x_{-1}, M(x_{-1}))
\]

• \( M(x_{-1}) \) is single valued means that only one type \( \theta \) is indifferent to \( (\hat{R}(x_{-1}), x_{-1}) \).

• This provides a condition for the marginal tax to exist and equal the tax wedge along the equilibrium set \( B \).

• If \( M(x_{-1}) \) is not single valued then we candidate MRSs...
  ...this actually implies kinks in \( \hat{R} \)
  ...we can still compute left and right derivatives

• for example: static Mirrlees (1971) when bunching occurs we get a convex kink in income tax schedule
1.5 Linear Taxes?

- can we choose a subset of goods to be taxed linearly? (not taxed is particular case, e.g. Atkinson-Stiglitz)

- suppose we can divide goods \( x = (x^a, x^b) \) so that

\[
\hat{x}^b(\theta) = \hat{x}^b(\theta') \implies \hat{x}^a(\theta) = \hat{x}^a(\theta') \quad \forall \theta, \theta' \in \Theta^2
\]

i.e. \( x^b \) identifies \( x^a \), write

\[
x^a = \hat{\alpha}(x^b)
\]

- typically

\[
\dim \Theta = \dim X^b \leq \dim X
\]

so this can be done

- define support of \( x^b \)

\[
B^b \equiv \{ x^b \mid \exists \theta \in \Theta \quad x^b = \hat{x}^b(\theta) \}
\]

- now, for given \( x^b \) consider the set

\[
B(x^b) \equiv \{ x^a \mid U(x^a, x^b, \theta) \leq \delta(\theta) \quad \forall \theta \in \Theta \} = \{ x^a \mid (x^a, x^b) \in \bar{B} \}
\]

- given \( x^b \in B^b \) define a linear set

\[
B^L(x^b) = \{ x^b \mid q(x^b) \cdot (x^a - \hat{\alpha}(x^b)) \leq 0 \}
\]

for some consumer prices \( q \) which may depend on \( x^b \)

- note: \( \hat{\alpha}(x^b) \in B^L(x^b) \)

- “mixed taxation”...

\[
B = \{ x \mid x^a \in B^L(x^b) \quad \text{and} \quad x^b \in B^b \}
\]

- Question: can this implement \( \hat{x} \)?

- Yes, if and only if

\[
B^L(x^b) \subseteq B(x^b)
\]
• sufficient condition: holds if $[B(x^b)]^c$ is convex

• in terms of taxes:
  \[ p \cdot x + T(x^a, x^b) \leq 0 \]
given $x^b$ can we make $T(\cdot, x^b)$ linear? i.e.
  \[ T(x^a, x^b) = t(x^b) + \tau(x^b) \cdot x^a \]

• sufficient condition: if $\bar{T}(\cdot, x^b)$ is convex then we use linear tangent

• Example: two-period consumption, linear tax on savings that depends on income

• with finite types and binding IC constraints:
  1. kinks! linear tax not possible
  2. but as types are closer: kinks get smaller
  3. near optimal allocation do not require kinks: linear tax possible

• with continuum of types: possible

1.6 Interdependence of Taxation

• note the tradeoff: linear tax but dependent on $x^b$

• sometimes possible to separate taxes...
  \[ T(x^a, x^b) = t^b(x^b) + t^a(x^a) \]

• Example: consumption two periods, nonlinear tax on income and savings (Estate Taxation paper Farhi-Werning)

2 Uncertainty

• opens many possibilities...
  general implementation: a dynamic choice problem

• Today: less general

• only uncertainty is $\theta_1$ at $t = 1$
– pre-committed goods $z$ (scalar; to simplify)
– ex-post goods $x(\theta)$ (vector)

- Resource constraint:
  \[ z + \frac{1}{R} p_x \cdot \int x(\theta) dF(\theta) \leq e \]
  with first element being numeraire: $p_{x,1} = 1$

- Utility
  \[ \mathbb{E}[U(z, x, \theta)] = \int U(z, x(\theta), \theta) dF(\theta) \]

- Example: two period Inverse euler example $z = c_0$ and $x = (c_1, y_1)$
  \[ U(c_0, (c_1, y_1), \theta) = u(c_0) + \beta u(c_1) − h(y_1; \theta) \]

- we take as given allocation $\hat{z}$ and $\hat{x}(\theta)$ and try to implement it

- Intertemporal wedge
  \[ (1 + \tau) \mathbb{E}[U_z(\hat{z}, \hat{x}(\theta), \theta)] = R \mathbb{E}[U_{x_1}(\hat{z}, \hat{x}(\theta), \theta)] \]

- Incentive compatibility...
  \[ U(\hat{z}, \hat{x}(\theta), \theta) \geq U(\hat{z}, \hat{x}(\theta'), \theta) \quad \theta', \theta \in \Theta^2 \]

- Budget constraint
  \[ z + s + T^z(s) \leq \hat{z} \]
  \[ p_x \cdot x + T^x(x_{-1}) \leq Rs \]

- note that $T^z$ does not depend on $\theta$

- we want to implement $s = 0$ (by “Ricardian equivalence” we could also do things with for any $s \neq 0$)

- Define $T^x(x_{-1}) \equiv -R^x_1(x_{-1}) - p_{-1} \cdot x_{-1}$
  \[ R^x(x_{-1}) \equiv \min_{\theta} U^{-1}(x_{-1}, \hat{z}, \theta) \]
• utility given this is...

\[ v(z, s, \theta) \equiv \max_x U(z, x, \theta) \]

s.t. \[ px \cdot x + T^x(x_{-1}) \leq Rs \]

• This function \( v(z, s, \theta) \) is continuos and differentiable in regions where the maximum is unique

• the Envelope condition at the proposed solution...

\[ v_z(\hat{z}, 0, \theta) = U_z(\hat{z}, \hat{x}(\theta), \theta) \]

\[ v_s(z, 0, \theta) = U_{x_1}(\hat{z}, \hat{x}(\theta), \theta) \]

• expected utility is

\[ V(z, s) \equiv \int v(z, x, \theta) dF(\theta) \]

• this function shares properties with \( v \); it may be smoother even due to the averaging across \( \theta \)...

• ...if \( \theta \) is continouosly distributed, \( \frac{\partial}{\partial s} V(z, s) \) and \( \frac{\partial}{\partial z} V(z, s) \) exist and

\[ \frac{\partial}{\partial s} V(z, s) = \int v_s(z, s, \theta) dF(\theta) \]

\[ \frac{\partial}{\partial z} V(z, s) = \int v_z(z, s, \theta) dF(\theta) \]

since the countable kinks in \( v \) do not matter when we average

• Now at \( t = 0 \) we want \( (z, s) = (\hat{z}, 0) \) so that

\[ \tilde{B}^z(z, s) = \{(z, s) \mid V(z, s) \leq V(\hat{z}, 0)\} \]

defines the largest set of pairs \( (z, s) \) that can be offered. Then

\[ V(z, s) = V(\hat{z}, 0) \]

defines the frontier of this set. In terms of taxes

\[ V(\hat{z} - s - T^s(s), s) = V(\hat{z}, 0) \]
Differentiating the definition of $T$ at equilibrium then gives

$\left(1 + \frac{\partial}{\partial s} T^s(s)\right) \mathbb{E}[U_{z_1}(\hat{z}, \hat{x}(\theta), \theta)] = \mathbb{E}[U_{x_1}(\hat{z}, \hat{x}(\theta), \theta)]$

if $F(\theta)$ is not continuous then we may have kinks in $T^s$

### 2.1 Alternative: State Contingent Linear Taxes

- separable utility case
- Kocherlakota proposes state dependent taxes
- define state dependent wedges:

$$U_{z_1}(\hat{z}, \hat{x}(\theta'), \theta) = (1 - \tau(\theta')) RU_{x_1}(\hat{z}, \hat{x}(\theta'), \theta)$$

with separability only depends on $\theta'$

- Budget constraint then

$$z + s \leq \hat{z}$$

$$x_1(\theta') = (1 - \tau(\theta')) Rs + \hat{x}_1(\theta')$$

$$x_{-1}(\theta') = \hat{x}_{-1}(\theta')$$

- note: we can turn this into

$$z + s \leq \hat{z}$$

$$x_1 + p \cdot x_{-1} + T(x_{-1}) = (1 - \tau(x_{-1})) Rs$$