Problem Set 6: Solutions

1. Let us draw the region where \( f_{X,Y}(x,y) \) is nonzero:

\[ y^2 - x = z \]

The joint PDF has to integrate to 1. From \( \int_{x=1}^{x=2} \int_{y=0}^{y=x} ax \, dy \, dx = \frac{7}{3}a = 1 \), we get \( a = \frac{3}{7} \).

(b) \( f_Y(y) = \int f_{X,Y}(x,y) \, dx = \begin{cases} \int_{x=1}^{x=2} \frac{3}{7}x \, dx, & \text{if } 0 \leq y \leq 1, \\ \int_{y}^{2} \frac{3}{7}x \, dx, & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{9}{14}, & \text{if } 0 \leq y \leq 1, \\ \frac{3}{14}(4 - y^2), & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise} \end{cases} \)

(c) \( f_{X|Y}(x \mid \frac{3}{2}) = \frac{f_{X,Y}(x, \frac{3}{2})}{f_Y(\frac{3}{2})} = \frac{8}{7}x, \quad \text{for } \frac{3}{2} \leq x \leq 2 \text{ and } 0 \text{ otherwise} \).

Then,

\[ E \left[ \frac{1}{X} \mid Y = \frac{3}{2} \right] = \int_{3/2}^{2} \frac{1}{x} \cdot \frac{1}{y} \, dx = \frac{4}{7} \]

(d) We use the technique of first finding the CDF and differentiating it to get the PDF.

\[ F_Z(z) = \begin{cases} 0, & \text{if } z < -2, \\ \int_{x=1}^{x=2} \int_{y=0}^{y=x+z} \frac{3}{7}x \, dy \, dx = \frac{8}{7} + \frac{6}{7}z - \frac{1}{14}z^3, & \text{if } -2 \leq z \leq -1, \\ \int_{x=1}^{x=2} \int_{y=0}^{y=x+z} \frac{3}{7}x \, dy \, dx = 1 + \frac{9}{14}z, & \text{if } -1 \leq z \leq 0, \\ 1, & \text{if } 0 < z. \end{cases} \]

\[ f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{6}{7} - \frac{3}{14}z^2, & \text{if } -2 \leq z \leq -1, \\ \frac{9}{14}, & \text{if } -1 < z \leq 0, \\ 0, & \text{otherwise}. \end{cases} \]
2. The PDF of $Z$, $f_Z(z)$, can be readily computed using the convolution integral:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t) f_Y(z - t) \, dt.$$  

For $z \in [-1, 0]$,

$$f_Z(z) = \int_{-1}^{z} \frac{1}{3} \cdot \frac{3}{4} (1 - t^2) \, dt = \frac{1}{4} \left( z - \frac{z^3}{3} + \frac{2}{3} \right).$$  

For $z \in [0, 1]$,

$$f_Z(z) = \int_{z-1}^{z} \frac{1}{3} \cdot \frac{3}{4} (1 - t^2) \, dt = \frac{1}{4} \left( 1 - \frac{z^3}{3} + \frac{(z-1)^3}{3} \right).$$

For $z \in [1, 2]$,

$$f_Z(z) = \int_{z-1}^{1} \frac{1}{3} \cdot \frac{3}{4} (1 - t^2) \, dt + \int_{-1}^{z-2} \frac{2}{3} \cdot \frac{3}{4} (1 - t^2) \, dt = \frac{1}{4} \left( z + \frac{(z-1)^3}{3} - \frac{2(z-2)^3}{3} - 1 \right).$$

For $z \in [2, 3]$,

$$f_Z(z) = \int_{z-3}^{z-2} \frac{2}{3} \cdot \frac{3}{4} (1 - t^2) \, dt = \frac{1}{6} (3 + (z-3)^3 - (z-2)^3).$$

For $z \in [3, 4]$,

$$f_Z(z) = \int_{z-3}^{1} \frac{2}{3} \cdot \frac{3}{4} (1 - t^2) \, dt = \frac{1}{6} \left( 11 - 3z + (z-3)^3 \right).$$

A sketch of $f_Z(z)$ is provided below.

3. (a) $X_1$ and $X_2$ are negatively correlated. Intuitively, a large number of tosses that result in a 1 suggests a smaller number of tosses that result in a 2.

(b) Let $A_t$ (respectively, $B_t$) be a Bernoulli random variable that is equal to 1 if and only if the $t$th toss resulted in 1 (respectively, 2). We have $E[A_t B_t] = 0$ (since $A_t \neq 0$ implies $B_t = 0$) and

$$E[A_t B_s] = E[A_t] E[B_s] = \frac{1}{k} \cdot \frac{1}{k} \quad \text{for} \quad s \neq t.$$  

Thus,

$$E[X_1 X_2] = E[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)] = n E[A_1(B_1 + \cdots + B_n)] = n(n-1) \cdot \frac{1}{k} \cdot \frac{1}{k}.$$
and
\[
\text{cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2]
\]
\[
= \frac{n(n-1)}{k^2} - \frac{n^2}{k^2} = -\frac{n}{k^2}.
\]

The covariance of $X_1$ and $X_2$ is negative as expected.

4. (a) If $X$ takes a value $x$ between $-1$ and 1, the conditional PDF of $Y$ is uniform between $-2$ and 2. If $X$ takes a value $x$ between 1 and 2, the conditional PDF of $Y$ is uniform between $-1$ and 1.

Similarly, if $Y$ takes a value $y$ between $-1$ and 1, the conditional PDF of $X$ is uniform between $-1$ and 2. If $Y$ takes a value $y$ between 1 and 2, or between $-2$ and $-1$, the conditional PDF of $X$ is uniform between $-1$ and 1.

(b) We have
\[
E[X | Y = y] = \begin{cases} 
0, & \text{if } -2 \leq y \leq -1, \\
1/2, & \text{if } -1 < y \leq 1, \\
0, & \text{if } 1 < y \leq 2, 
\end{cases}
\]
and
\[
\text{var}(X | Y = y) = \begin{cases} 
1/3, & \text{if } -2 \leq y \leq -1, \\
3/4, & \text{if } -1 < y \leq 1, \\
1/3, & \text{if } 1 \leq y \leq 2. 
\end{cases}
\]

It follows that $E[X] = 3/10$ and $\text{var}(X) = 193/300$.

(c) By symmetry, we have $E[Y | X] = 0$ and $E[Y] = 0$. Furthermore, $\text{var}(Y | X = x)$ is the variance of a uniform PDF (whose range depends on $x$), and
\[
\text{var}(Y | X = x) = \begin{cases} 
4/3, & \text{if } -1 \leq x \leq 1, \\
1/3, & \text{if } 1 < x \leq 2. 
\end{cases}
\]

Using the law of total variance, we obtain
\[
\text{var}(Y) = E[\text{var}(Y | X)] = \frac{4}{5} \cdot \frac{4}{3} + \frac{1}{5} \cdot \frac{1}{3} = 17/15.
\]

5. First let us write out the properties of all of our random variables. Let us also define $K$ to be the number of members attending a meeting and $B$ to be the Bernoulli random variable describing whether or not a member attends a meeting.

\[
E[N] = \frac{1}{1-p}, \quad \text{var}(N) = \frac{p}{(1-p)^2},
\]
\[
E[M] = \frac{1}{\lambda}, \quad \text{var}(M) = \frac{1}{\lambda^2},
\]
\[
E[B] = q, \quad \text{var}(B) = q(1-q).
\]

(a) Since $K = B_1 + B_2 + \cdots + B_N$,
\[
E[K] = E[N] \cdot E[B] = \frac{q}{1-p},
\]
\[
\text{var}(K) = E[N] \cdot \text{var}(B) + (E(B))^2 \cdot \text{var}(N) = \frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}.
\]
G1†. (a) Let $X_1, X_2, \ldots , X_n$ be independent, identically distributed (IID) random variables. We note that

$$E[X_1 + \cdots + X_n | X_1 + \cdots + X_n = x_0] = x_0.$$ 

It follows from the linearity of expectations that

$$x_0 = E[X_1 + \cdots + X_n | X_1 + \cdots + X_n = x_0] = E[X_1 | X_1 + \cdots + X_n = x_0] + \cdots + E[X_n | X_1 + \cdots + X_n = x_0].$$ 

Because the $X_i$’s are identically distributed, we have the following relationship.

$$E[X_i | X_1 + \cdots + X_n = x_0] = E[X_j | X_1 + \cdots + X_n = x_0],$$

for any $1 \leq i \leq n, 1 \leq j \leq n.$

Therefore,

$$nE[X_1 | X_1 + \cdots + X_n = x_0] = x_0$$

$$E[X_1 | X_1 + \cdots + X_n = x_0] = \frac{x_0}{n}.$$

(b) Note that we can rewrite $E[X_1 | S_n = s_n, S_{n+1} = s_{n+1}, \ldots , S_{2n} = s_{2n}]$ as follows:

$$E[X_1 | S_n = s_n, S_{n+1} = s_{n+1}, \ldots , S_{2n} = s_{2n}] = E[X_1 | S_n = s_n, X_{n+1} = s_{n+1} - s_n, X_{n+2} = s_{n+2} - s_{n+1}, \ldots , X_{2n} = s_{2n} - s_{2n-1}].$$

where the last equality holds due to the fact that the $X_i$’s are independent. We also note that

$$E[X_1 + \cdots + X_n | S_n = s_n] = E[S_n | S_n = s_n] = s_n.$$ 

It follows from the linearity of expectations that

$$E[X_1 + \cdots + X_n | S_n = s_n] = E[X_1 | S_n = s_n] + \cdots + E[X_n | S_n = s_n].$$

Because the $X_i$’s are identically distributed, we have the following relationship:

$$E[X_i | S_n = s_n] = E[X_j | S_n = s_n],$$

for any $1 \leq i \leq n, 1 \leq j \leq n.$

Therefore,

$$E[X_1 + \cdots + X_n | S_n = s_n] = nE[X_1 | S_n = s_n] = s_n \Rightarrow E[X_1 | S_n = s_n] = \frac{s_n}{n}.$$