LECTURE 17
Markov Processes – II

• Readings: Section 7.3

Lecture outline
• Review
• Steady-State behavior
  • Steady-state convergence theorem
  • Balance equations
• Birth-death processes

Review
• Discrete state, discrete time, time-homogeneous
  • Transition probabilities $p_{ij}$
  • Markov property

• $r_{ij}(n) = P(X_n = j \mid X_0 = i)$

• Key recursion:
  $r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$

Warmup

\[ P(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) = \]

\[ P(X_4 = 7 \mid X_0 = 2) = \]

**Recurrent and transient states**

• State $i$ is **recurrent** if:
  starting from $i$,
  and from wherever you can go,
  there is a way of returning to $i$

• If not recurrent, called **transient**

• **Recurrent class**:
  collection of recurrent states that
  “communicate” to each other
  and to no other state

Periodic states

• The states in a recurrent class are **periodic** if they can be grouped into
  $d > 1$ groups so that all transitions from
  one group lead to the next group
Steady-State Probabilities

- Do the $r_{ij}(n)$ converge to some $\pi_j$? (independent of the initial state $i$)
- Yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic
- Assuming “yes,” start from key recursion
  $$r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$$
  - take the limit as $n \to \infty$
  $$\pi_j = \sum_k \pi_k p_{kj}$$
  - Additional equation:
  $$\sum_j \pi_j = 1$$

Visit frequency interpretation

$$\pi_j = \sum_k \pi_k p_{kj}$$

- (Long run) frequency of being in $j$: $\pi_j$
- Frequency of transitions $k \to j$: $\pi_k p_{kj}$
- Frequency of transitions into $j$: $\sum_k \pi_k p_{kj}$

Example Birth-death processes

$$\pi_i p_i = \pi_{i+1} q_{i+1}$$

- Special case: $p_i = p$ and $q_i = q$ for all $i$
  - $\rho = p/q$ = load factor
  $$\pi_{i+1} = \frac{\pi_i p}{q} = \pi_i \rho$$
  $$\pi_i = \pi_0 \rho^i, \quad i = 0, 1, \ldots, m$$
- Assume $p < q$ and $m \approx \infty$
  $$\pi_0 = 1 - \rho$$
  $$E[X_0] = \frac{\rho}{1 - \rho} \quad \text{(in steady-state)}$$
Fall 2010

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