Lecture 4

Readings: Section 1.6

Lecture outline

- Principles of counting
- Many examples
  - permutations
  - k-permutations
  - combinations
  - partitions
- Binomial probabilities

Discrete uniform law

- Let all sample points be equally likely
- Then,
  \[ P(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}} = \frac{|A|}{|\Omega|} \]
- Just count...

Basic counting principle

- r stages
- n_i choices at stage i

- Number of choices is: \( n_1 n_2 \cdots n_r \)

- Number of license plates with 3 letters and 4 digits =

- ... if repetition is prohibited =

- Permutations: Number of ways of ordering n elements is:

- Number of subsets of \{1, \ldots, n\} =

Example

- Probability that six rolls of a six-sided die all give different numbers?
  - Number of outcomes that make the event happen:
  - Number of elements in the sample space:
  - Answer:
Combinations

- \( \binom{n}{k} \): number of \( k \)-element subsets of a given \( n \)-element set
- Two ways of constructing an ordered sequence of \( k \) distinct items:
  - Choose the \( k \) items one at a time:
    \[
    n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!} \text{ choices}
    \]
  - Choose \( k \) items, then order them \((k!) \text{ possible orders}\)
- Hence:
  \[
  \binom{n}{k} \cdot k! = \frac{n!}{(n-k)!}
  \]
  \[
  \binom{n}{k} = \frac{n!}{k!(n-k)!}
  \]
  \[
  \sum_{k=0}^{n} \binom{n}{k} = 2^n
  \]

Binomial probabilities

- \( n \) independent coin tosses
  - \( P(H) = p \)
  - \( P(HTTHHH) = \)
  - \( P(\text{sequence}) = p^\# \text{ heads} (1-p)^\# \text{ tails} \)
  - \( P(k \text{ heads}) = \sum_{k=\text{head seq.}} P(\text{seq.}) \)
    \[
    = (\# \text{ of } k \text{-head seqs}) \cdot p^k(1-p)^{n-k}
    \]
    \[
    = \left( \binom{n}{k} \right) p^k(1-p)^{n-k}
    \]

Coin tossing problem

- event \( B \): 3 out of 10 tosses were “heads”.
  - Given that \( B \) occurred, what is the (conditional) probability that the first 2 tosses were heads?
- All outcomes in set \( B \) are equally likely: probability \( p^3(1-p)^7 \)
- Conditional probability law is uniform
- Number of outcomes in \( B \):
- Out of the outcomes in \( B \), how many start with HH?

Partitions

- 52-card deck, dealt to 4 players
  - Find \( P(\text{each gets an ace}) \)
  - Outcome: a partition of the 52 cards
    - number of outcomes:
    \[
    \frac{52!}{13!13!13!13!}
    \]
  - Count number of ways of distributing the four aces: \( 4 \cdot 3 \cdot 2 \)
  - Count number of ways of dealing the remaining 48 cards
    \[
    \frac{48!}{12!12!12!12!}
    \]
  - Answer:
    \[
    \frac{4 \cdot 3 \cdot 2 \cdot 48!}{12!12!12!12!} = \frac{52!}{13!13!13!13!}
    \]