Problem Set 3 Solutions

Problem 3.1

Consider a control system described by

\[ \ddot{q}(t) - a^2 q(t) = v(t) + f_1(t), \quad g(t) = q(t) + f_2(t), \]

where \( f = [f_1; f_2] \) is a normalized white noise, \( v \) is the control signal, \( g \) is the sensor measurement, and \( a > 0 \) is a parameter. The objective is to find a dynamic feedback controller (with input \( g \) and output \( v \)) which stabilizes the system while using a minimum of control effort (defined as the asymptotic variance of \( v(t) \) as \( t \to \infty \)).

(a) Find the coefficients of the auxiliary abstract H2 optimization problems associated with the original task.

For

\[ x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad w = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad u = v, \quad y = g, \quad z = v, \]

we have

\[ A = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]
\[ C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \]

\({}^1\text{Version of March 9, 2004}\)
$D_{12} = 1, \ D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D_{22} = 0.$

The full information control abstract H2 optimization has coefficients

$$a = A, \ b = B_2, \ c = C_1, \ d = D_{12}.$$ 

The state estimation abstract H2 optimization problem has coefficients

$$a = A', \ b = C'_2, \ c = B'_1, \ d = D'_{21}.$$ 

(b) Write analytically the associated Hamiltonian matrices, bases of their stable invariant subspaces, stabilizing solutions of the Riccati equations, and optimal controller and observer gains.

The full information control Hamiltonian is

$$\mathcal{H}_{fi} = \begin{bmatrix} A & B_2B'_2 \\ 0 & -A' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a^2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -a^2 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$ 

Its eigenvalues are $\pm a$ (double multiplicity each). Hence the stable invariant subspace of $\mathcal{H}_{fi}$ is the kernel of $(\mathcal{H}_{fi} + aI)^2$. A basis in this kernel is given by

$$\begin{bmatrix} 1 \\ 0 \\ -2a^3 \\ -2a^2 \end{bmatrix}, \ \begin{bmatrix} -1/a \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

Hence

$$P_{fi} = -\begin{bmatrix} -2a^3 & 0 \\ -2a^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1/a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2a^3 & 2a^2 \\ 2a^2 & 2a \end{bmatrix},$$

and the optimal state feedback gain is given by

$$K = -B'_2P_{fi} = \begin{bmatrix} -2a^2 & -2a \end{bmatrix}.$$ 

A good sanity check here: the closed loop poles (eigenvalues of $A + B_2K$) should be identical to the stable eigenvalues of the Hamiltonian.

The state estimation Hamiltonian is

$$\mathcal{H}_{se} = \begin{bmatrix} A' & C'_2C_2 \\ B_1B'_1 & -A \end{bmatrix} = \begin{bmatrix} 0 & a^2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -a^2 & 0 \end{bmatrix}.$$
Its characteristic polynomial is \( s^4 - 2a^2s^2 + a^4 + 1 \), and hence its eigenvalues \( s \) satisfy \( s^2 = a^2 \pm j \). The stable eigenvalue \( s = x + jy \) such that \( s^2 = a^2 + j \) is given by

\[
x = -\sqrt{\frac{\sqrt{a^4 + 1} + a^2}{2}}, \quad y = -\sqrt{\frac{\sqrt{a^4 + 1} - a^2}{2}}.
\]

The corresponding eigenvector is

\[
h = \begin{bmatrix} x + jy \\ 1 \\ j \\ y - jx \end{bmatrix}.
\]

Since \( \mathcal{H}_{se} \) has real coefficients, the eigenvector corresponding to eigenvalue \( x - jy \) will be the complex conjugate of \( h \). Hence real and imaginary parts of \( h \) form a basis in the stable invariant subspace of \( \mathcal{H}_{se} \):

\[
\begin{bmatrix} y \\ 0 \\ 1 \\ -x \end{bmatrix}, \quad \begin{bmatrix} x \\ 1 \\ 0 \\ y \end{bmatrix}.
\]

Hence

\[
P_{se} = -\begin{bmatrix} 1 & 0 \\ -x & y \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/y & x/y \\ x/y & -x^2/y - y \end{bmatrix},
\]

and

\[
L = P_{se}C_2' = \begin{bmatrix} 1/y \\ -x/y \end{bmatrix}.
\]

(c) Derive an analytical expression for the transfer function of the optimal dynamic feedback controller, and verify it using numerical calculations with h2syn.m.

The closed loop transfer function is given by

\[
G(s) = -K(sI - A - B_2K - LC_2)^{-1}L.
\]

To verify the formulae, MATLAB function ps3_1.m can be used. This function relies on SIMULINK design diagram ps3_1a.m.
Problem 3.2

Random signal $q = q(t)$ is assumed to be a “bandlimited white noise” of a given bandwidth $B$ (i.e. the result of passing the true white noise $v_1(t)$ through an ideal low-pass filter of bandwidth $B$ rad/sec). A high quality sensor is assumed to measure $q(t)$ accurately, except for a white additive noise, with the signal-to-noise ratio of 10.

(a) Use $\text{h2syn.m}$ to design a 10-th order linear filter which inputs the sensor output, and outputs an estimate of $\dot{q}(t)$ which makes the mean square estimation error as small as possible.

M-function $\text{ps3.2a.m}$ does the job. It uses a 10-th order Butterworth filter $W$ with cut-off frequency $B$ to model $q$ as $q = Ww$, where $w$ is the normalized white noise.

(b) Test your design by comparing the simulated performance of filters you have designed for $B = 10$ rad/sec and $B = 1$ rad/sec on signals $q(\cdot)$ of bandwidths of $B = 10$ and $B = 1$ rad/sec. (One expects that the filter optimized for $B = 10$ rad/sec will be better on the $q(\cdot)$ with bandwidth $B = 10$ rad/sec than the filter optimized for $B = 1$ rad/sec, and vice versa.) Use the generator of bandlimited white noise supplied with the SIMULINK to perform the simulations.

The SIMULINK diagram for testing is $\text{ps3.2c.mdl}$. You must run $\text{ps3.2b.m}$ before you open it. For $B = 1$ rad/sec, the degradation of performance when a filter designed for $B = 10$ is used is dramatic (at least a 10-fold increase of error). For $B = 10$, the degradation of performance when a filter designed for $B = 1$ is used is not as big, but still quite noticeable.