Problem 1
(a) Let \( C \) be a nonempty convex cone. Show that \( cl(C) \) and \( ri(C) \) is also a convex cone.
(b) Let \( C = cone\{x_1, \ldots, x_m\} \). Show that
\[
ri(C) = \{ \sum_{i=1}^{m} a_i x_i | a_i > 0, i = 1, \ldots, m \}.
\]

Problem 2
Let \( C_1 \) and \( C_2 \) be convex sets. Show that
\[
C_1 \cap ri(C_2) \neq \emptyset \quad \text{if and only if} \quad ri(C_1 \cap aff(C_2)) \cap ri(C_2) \neq \emptyset.
\]

Problem 3
(a) Consider a vector \( x^* \) such that a given function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex over a sphere centered at \( x^* \). Show that \( x^* \) is a local minimum of \( f \) if and only if it is a local minimum of \( f \) along every line passing through \( x^* \) [i.e., for all \( d \in \mathbb{R}^n \), the function \( g : \mathbb{R} \rightarrow \mathbb{R} \), defined by \( g(\alpha) = f(x^* + \alpha d) \), has \( \alpha^* = 0 \) as its local minimum].
(b) Consider the nonconvex function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by
\[
f(x_1, x_2) = (x_2 - px_1^2)(x_2 - qx_1^2),
\]
where \( p \) and \( q \) are scalars with \( 0 < p < q \), and \( x^* = (0, 0) \). Show that \( f(y, my^2) < 0 \) for \( y \neq 0 \) and \( m \) satisfying \( p < m < q \), so \( x^* \) is not a local minimum of \( f \) even though it is a local minimum along every line passing through \( x^* \).

Problem 4
(a) Consider the quadratic program
\[
\begin{align*}
& \text{minimize} & & \frac{1}{2} |x|^2 + c'x \\
& \text{subject to} & & Ax = 0
\end{align*}
\]
where \( c \in \mathbb{R}^n \) and \( A \) is an \( m \times n \) matrix of rank \( m \). Use the Projection Theorem to show that
\[
x^* = - (I - A'(AA')^{-1}A)c
\]
is the unique solution.
(b) Consider the more general quadratic program

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \left( x - \bar{x} \right)' Q (x - \bar{x}) + c'(x - \bar{x}) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

where \( c \) and \( A \) are as before, \( Q \) is a symmetric positive definite matrix, \( b \in \mathbb{R}^m \), and \( \bar{x} \) is a vector in \( \mathbb{R}^n \), which is feasible, i.e., satisfies \( A\bar{x} = b \). Use the transformation \( y = Q^{1/2}(x - \bar{x}) \) to write this problem in the form of part (a) and show that the optimal solution is

\[
x^* = \bar{x} - Q^{-1}(c - A'\lambda),
\]

where \( \lambda \) is given by

\[
\lambda = (AQ^{-1}A')^{-1}AQ^{-1}c.
\]

(c) Apply the result of part (b) to the program

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x'Qx + c'x \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

and show that the optimal solution is

\[
x^* = -Q^{-1}(c - A'\lambda - A'(AQ^{-1}A')^{-1}b).
\]

Problem 5

Let \( X \) be a closed convex subset of \( \mathbb{R}^n \), and let \( f : \mathbb{R}^n \mapsto (-\infty, \infty] \) be a closed convex function such that \( X \cap \text{dom}(f) \neq \emptyset \). Assume that \( f \) and \( X \) have no common nonzero direction of recession. Let \( X^* \) be the set of minima of \( f \) over \( X \) (which is nonempty and compact), and let \( f^* = \inf_{x \in X} f(x) \). Show that:

(a) For every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that every vector \( x \in X \) with \( f(x) \leq f^* + \delta \) satisfies \( \min_{x^* \in X^*} \| x - x^* \| \leq \epsilon \).

(b) If \( f \) is real-valued, for every \( \delta > 0 \) there exists an \( \epsilon > 0 \) such that every vector \( x \in X \) with \( \min_{x^* \in X^*} \| x - x^* \| \leq \epsilon \) satisfies \( f(x) \leq f^* + \delta \).

(c) Every sequence \( \{ x_k \} \subset X \) satisfying \( f(x_k) \to f^* \) is bounded and all its limit points belong to \( X^* \).