Introduction to Simulation - Lecture 19

Laplace’s Equation – FEM Methods

Jacob White

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Outline for Poisson Equation Section

- Why Study Poisson’s equation
  - Heat Flow, Potential Flow, Electrostatics
  - Raises many issues common to solving PDEs.
- Basic Numerical Techniques
  - basis functions (FEM) and finite-differences
  - Integral equation methods
- Fast Methods for 3-D
  - Preconditioners for FEM and Finite-differences
  - Fast multipole techniques for integral equations
Outline for Today

- Why Poisson Equation
  - Reminder about heat conducting bar
- Finite-Difference And Basis function methods
  - Key question of convergence
- Convergence of Finite-Element methods
  - Key idea: solve Poisson by minimization
  - Demonstrate optimality in a carefully chosen norm
Drag Force Analysis of Aircraft

- Potential Flow Equations
  - Poisson Partial Differential Equations.
Engine Thermal Analysis

- Thermal Conduction Equations
  - The Poisson Partial Differential Equation.
Capacitance on a microprocessor Signal Line

- Electrostatic Analysis
  - The Laplace Partial Differential Equation.
Question: What is the temperature distribution along the bar?
1) Cut the bar into short sections

2) Assign each cut a temperature
Heat Flow through one section

\[ h_{i+1,i} = \text{heat flow} = \kappa \frac{T_{i+1} - T_i}{\Delta x} \]

Limit as the sections become vanishingly small

\[ \lim_{\Delta x \to 0} h(x) = \kappa \frac{\partial T(x)}{\partial x} \]
Heat Flows into Control Volume Sums to zero

\[ h_{i+1,i} - h_{i,i-1} = -h_s \Delta x \]
Heat Flows into Control Volume Sums to zero

\[ h_{i+1,i} - h_{i,i-1} = -h_s \Delta x \]

- **Incoming Heat (\(h_i\))**
- **Heat in from left**
- **Heat out from right**

**Limit as the sections become vanishingly small**

\[
\lim_{\Delta x \to 0} h_s (x) = \frac{\partial h(x)}{\partial x} = \frac{\partial}{\partial x} \kappa \frac{\partial T(x)}{\partial x}
\]
Temperature analogous to Voltage
Heat Flow analogous to Current

\[
\frac{1}{R} = \frac{\kappa}{\Delta x}
\]

\[
T_1 \quad T_N
\]

\[
v_s = T(0) \quad i_s = h_s \Delta x
\]

\[
v_s = T(1)
\]
Normalized Poisson Equation

\[ \frac{\partial}{\partial x} \kappa \frac{\partial T(x)}{\partial x} = -h_s \Rightarrow -\frac{\partial^2 u(x)}{\partial x^2} = f(x) \]

\[ -u_{xx}(x) = f(x) \]
Numerical Solution

Finite Differences

Discretization

Subdivide interval \((0, 1)\) into \(n + 1\) equal subintervals

\[
\Delta x = \frac{1}{n + 1}
\]

\[
x_j = j \Delta x, \quad \hat{u}_j \approx u_j \equiv u(x_j)
\]

for \(0 \leq j \leq n + 1\)
Numerical Solution

For example . . .

\[ v''(x_j) \approx \frac{1}{\Delta x} (v'(x_{j+1/2}) - v'(x_{j-1/2})) \]

\[ \approx \frac{1}{\Delta x} \left( \frac{v_{j+1} - v_j}{\Delta x} - \frac{v_j - v_{j-1}}{\Delta x} \right) \]

\[ = \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2} \]

for \( \Delta x \) small
Using Basis Functions

Residual Equation

Partial Differential Equation form

\[ -\frac{\partial^2 u}{\partial x^2} = f \quad u(0) = 0 \quad u(1) = 0 \]

Basis Function Representation

\[ u(x) \approx u_h(x) = \sum_{i=1}^{n} \omega_i \phi_i(x) \]

Plug Basis Function Representation into the Equation

\[ R(x) = \sum_{i=1}^{n} \omega_i \frac{d^2 \phi_i(x)}{dx^2} + f(x) \]
Using Basis Functions

Introduce basis representation \( u(x) \approx u_h(x) = \sum_{i=1}^{n} \omega_i \phi_i(x) \)

\( \Rightarrow u_h(x) \) is a weighted sum of basis functions

The basis functions define a space

\[ X_h = \left\{ v \in X_h \mid v = \sum_{i=1}^{n} \beta_i \phi_i \text{ for some } \beta_i \text{'s} \right\} \]

**Example**

“Hat” basis functions

Piecewise linear Space
Using Basis functions

Basis Weights

Galerkin Scheme

Force the residual to be “orthogonal” to the basis functions

\[ \int_{0}^{1} \varphi_{l}(x) R(x) \, dt = 0 \]

Generates \( n \) equations in \( n \) unknowns

\[ \int_{0}^{1} \varphi_{l}(x) \left[ \sum_{i=1}^{n} \omega_{i} \frac{d^{2} \varphi_{i}(x)}{dx^{2}} + f(x) \right] \, dx = 0 \quad l \in \{1, \ldots, n\} \]
Using Basis Functions

Basis Weights
Galerkin with integration by parts

Only first derivatives of basis functions

\[
\int_{0}^{1} \frac{d \varphi_l(x)}{dx} \left( \sum_{i=1}^{n} \omega_i \varphi_i(x) \right) dx - \int_{0}^{1} \varphi_i(x) f(x) dx = 0
\]

\[
l \in \{1, \ldots, n\}
\]
The question is

How does $\|u - u_h\|$ decrease with refinement?

- This time – Finite-element methods
- Next time – Finite-difference methods
Heat Equation

Partial Differential Equation form

\[- \frac{\partial^2 u}{\partial x^2} = f \quad u(0) = 0 \quad u(1) = 0\]

“Nearly” Equivalent weak form

\[\int_\Omega \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx = \int_\Omega f v dx \quad \text{for all } v\]

\[a(u, v) = l(v) \quad \text{for all } v\]
Convergence Analysis

Overview of FEM

Heat Equation

Introduce basis representation

\[ u(x) \approx u_h(x) = \sum_{i=1}^{n} \omega_i \phi_i(x) \]

⇒ \( u_h(x) \) is a weighted sum of basis functions

The basis functions define a space

\[ X_h = \left\{ \nu \in X_h \mid \nu = \sum_{i=1}^{n} \beta_i \phi_i \text{ for some } \beta_i's \right\} \]

Example

“Hat” basis functions

Piecewise linear Space

\[ \phi_1, \phi_2, \phi_4, \phi_6 \]

\[ \phi_3, \phi_5 \]
Key Idea

\[ a(u, u) \text{ defines a norm } a(u, u) = \| u \| \]

\( U \) is restricted to be 0 at 0 and 1!!

Using the norm properties, it is possible to show

\[ \text{If } a(u_h, \varphi_i) = l(\varphi_i) \text{ for all } \varphi_i \in \{ \varphi_1, \varphi_2, \ldots, \varphi_n \} \]

\[ \text{Then } \| u - u_h \| = \min_{w_h \in X_h} \| u - w_h \| \]

Solution Error

Projection Error
The question is only:

How well can you fit $u$ with a member of $X_h$?

But you must measure the error in the $||| \cdot |||$ norm.

For piecewise linear:

$$||| u - u_h ||| = O\left(\frac{1}{n}\right)$$
Boundary Value Problem (BVP) - Strong Form

$$-u''(x) + \alpha u(x) = f(x) \quad \alpha \geq 0$$

$$x \in (0, 1), \quad u(0) = u(1) = 0$$

Describes many physical phenomena (e.g.):

- Temperature distribution in a bar *
- Deformation of an elastic bar
- Deformation of a string under tension
The solution $u(x)$ always exists.

$u(x)$ is always smoother than the data $f(x)$.

Given $f(x)$, the solution $u(x)$ is unique.
Minimization Principle

Find

$$u = \arg \min_{w \in X} J(w)$$

where

$$X = \{v \text{ sufficiently smooth} \mid v(0) = v(1) = 0\},$$

and

$$J(w) = \frac{1}{2} \int_0^1 (w_x w_x + \alpha w w) \, dx - \int_0^1 f w \, dx$$
Minimization Principle

In words:

Over all functions \( w \) in \( X \),

\( u \) that satisfies

\[
-u_{xx} + \alpha u = f \quad \text{in } \Omega
\]

\( u(0) = u(1) = 0 \)

makes \( J(w) \) as small as possible.
Minimization Principle

Let \( w = u + v \).
Then

\[
J\left(\sum_{x \in X} \frac{w}{x} \right) = \frac{1}{2} \int_0^1 (u + v)_x(u + v)_x \, dx \\
+ \frac{\alpha}{2} \int_0^1 (u + v)(u + v) \, dx \\
- \int_0^1 f(u + v) \, dx .
\]
Minimization Principle

\[ J(u + v) = \frac{1}{2} \int_0^1 (u_x u_x + \alpha u u) \, dx - \int_0^1 f u \, dx \]

\[ J(u) \]

\[ + \int_0^1 (u_x v_x + \alpha u v) \, dx - \int_0^1 f v \, dx \]

\[ \delta J_v(u) \]

first variation

\[ + \frac{1}{2} \int_0^1 (v_x v_x + \alpha v v) \, dx \]

\[ > 0 \text{ for } v \neq 0 \]
Minimization Principle

\[ \delta J_v(u) = \int_0^1 (u_x v_x + \alpha uv) \, dx - \int_0^1 f v \, dx \]

\[ = \hat{\mathcal{J}}^0(0) u_x(0) - \hat{\mathcal{J}}^0(1) u_x(1) - \int_0^1 u_{xx} v \, dx \]

\[ + \alpha \int_0^1 u v \, dx - \int_0^1 f v \, dx \]

\[ = \int_0^1 v \left\{ -u_{xx} + \alpha u - f \right\} \, dx = 0, \ \forall v \in X \]
Minimization Principle

\[ J(u + v) = J(u) + \frac{1}{2} \int_0^1 (v_x v_x + \alpha v v) \, dx, \quad \forall v \in X \]

\[ > 0 \text{ unless } v = 0 \]

\[ J(w) > J(u), \quad \forall w \in X, \ w \neq u \]

\[ \implies \]

\[ u \text{ is the minimizer of } J(w) \]
Rayleigh-Ritz Approach

\[ x = 0 \quad \rightarrow \quad h \quad \leftarrow \quad x = 1 \]

\[ x_0 \quad T_h^1 \quad T_h^k \quad x_i \quad x_{n+1} \]

\[ \bar{\Omega} = \bigcup_{k=1}^{K} T_h^k \quad T_h^k, \ k = 1, \ldots, K = n + 1: \text{elements} \]

\[ x_i, \ i = 0, \ldots, n + 1: \text{nodes} \]
Rayleigh-Ritz Approach

Approximation

Space \( X_h \subset X \)

\[
X_h = \left\{ v \in X \mid v|_{T_h^k} \in \mathbb{P}_1(T_h^k), \quad k = 1, \ldots, K \right\}
\]

\( v \in X_h \) is piecewise linear

\( v(0) = 0 \)

\( v(1) = 0 \)

\( v \) is continuous
Nodal basis for $X_h$:

$\varphi_j, j = 1, \ldots, n = \dim(X_h)$

$\varphi_i$ nonzero only on $T_h^i \cup T_h^{i+1}$
Let

\[ u_h \in X_h = \sum_{j=1}^{n} u_{hj} \varphi_j(x) ;\]

set \( u_{hj} = w_j \) that minimize

\[ J \left( \sum_{j=1}^{n} w_j \varphi_j \right) . \]
Rayleigh-Ritz Approach

“Projection”

...Plan

Geometric Picture:

$J_{X_h}$

(minimizer over $X_h$)

(minimizer over $X$)
Rayleigh-Ritz Approach

\[ J \left( \sum_{j=1}^{n} w_j \varphi_j \right) = \frac{1}{2} \int_{0}^{1} \frac{d}{dx} \left( \sum_{i=1}^{n} w_i \varphi_i \right) \frac{d}{dx} \left( \sum_{j=1}^{n} w_j \varphi_j \right) \]

\[ + \frac{\alpha}{2} \int_{0}^{1} \sum_{i=1}^{n} (w_i \varphi_i) \sum_{j=1}^{n} (w_j \varphi_j) - \int_{0}^{1} f \sum_{j=1}^{n} w_j \varphi_j \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \int_{0}^{1} \left( \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} + \alpha \varphi_i \varphi_j \right) dx - \sum_{j=1}^{n} w_j \int_{0}^{1} f \varphi_j dx \]

by bilinearity and linearity.
Rayleigh-Ritz Approach

\[ J^R(\mathbf{w} \in \mathbb{R}^n) \equiv J \left( \sum_{j=1}^{n} w_j \varphi_j \right) \]

\[ = \frac{1}{2} \mathbf{w}^T A_h \mathbf{w} - \mathbf{w}^T F_h. \]

\[ F_h \in \mathbb{R}^n: F_{hi} = \int_0^1 f \varphi_i \, dx \]

\[ A_h \in \mathbb{R}^{n \times n}: A_{hij} = \int_0^1 \left( \frac{d \varphi_i}{dx} \frac{d \varphi_j}{dx} + \alpha \varphi_i \varphi_j \right) \, dx \]

"Projection"

...\[ J \mid x_h \]
Rayleigh-Ritz Approach

\[ u_h = \arg \min_{w \in \mathbb{R}^n} J^R(w) \]

Expand \( J(w = u_h + v) \); require \( J(w) > J(u_h) \) unless \( v = 0 \).
Rayleigh-Ritz Approach

\[ J^R(u_h + v) \]

\[ = \frac{1}{2} (u_h + v)^T A_h (u_h + v) - (u_h + v)^T F_h \]

\[ = \frac{1}{2} u_h^T A_h u_h - u_h^T F_h \]

\[ + \frac{1}{2} v^T A_h u_h + \frac{1}{2} u_h^T A_h v - v^T F_h \]

\[ + \frac{1}{2} v^T A_h v \]
Rayleigh-Ritz Approach

\[
J^R(u_h + v) = J(u) + \left( A_h u_h - F_h \right)^T \nabla J^R(u_h) v + \delta J^R_v(u_h)
\]

\[
+ \frac{1}{2} v^T A_h v \quad \text{SPD}
\]

...Minimization...

“Projection”
Rayleigh-Ritz Approach

If (and only if)

\[ \delta J^R_v(u_h) = 0, \quad \forall v \in \mathbb{R}^n \]

\[ \Leftrightarrow \]

\[ \nabla J^R(u_h) = A_h u_h - F_h = 0 \]

then

\[ J(w = u_h + v) > J(u_h), \quad \forall v \neq 0. \]
Rayleigh-Ritz Approach

Find \( \mathbf{u}_h \in \mathbb{R}^n \) such that

\[
A_h \, \mathbf{u}_h = F_h \quad \Rightarrow \quad \mathbf{u}_h(x) = \sum_{j=1}^{n} u_{hj} \varphi_j(x).
\]

SPD \( \Rightarrow \) existence and uniqueness.
Error Analysis

Remember

\[ J(u + v) = J(u) + \frac{1}{2} \int_0^1 (v_x v_x + \alpha v v) \, dx, \quad \forall v \in X \]

\[ \geq 0, SPD \]

Define

\[ |||v||| = \left[ \int_0^1 (v_x v_x + \alpha v v) \, dx \right]^{\frac{1}{2}} \]

Energy norm
Error Analysis

Therefore

\[ J(u + v) = J(u) + \frac{1}{2}|||v|||^2, \quad \forall v \in X \]

Choose any \( w_h \in X_h, \ v \rightarrow (w_h - u) \in X \)

\[ J(w_h) = J(u) + \frac{1}{2}|||u - w_h|||^2, \quad \forall w_h \in X_h \]

For \( w_h = u_h \)

\[ J(u_h) = J(u) + \frac{1}{2}|||u - u_h|||^2 \]
Error Analysis

\[ J(u_h) < J(w_h), \quad \forall w_h \in X_h, \ w_h \neq u_h \]

if \( e = u - u_h \)

\[ \|\|u - u_h\|\| < \|\|u - w_h\|\|, \quad \forall w_h \in X_h, \ w_h \neq u_h \]

and

\[ \|\|e\|\| = \inf_{w_h \in X_h} \|\|u - w_h\|\| \]
Error Analysis

Energy norm

\textit{In words:} even if you knew $u$,

you could not find a $w_h$ in $X_h$

more accurate than $u_h$

\textit{in the energy norm.}
Error Analysis

A priori error estimates

Energy norm:

$$\|\|e\|\| \leq C_1 h$$

$L_2$ norm:

$$\|e\| = \left( \int_0^1 e e \, dx \right)^{1/2} \leq C_2 h^2$$

$$C_{1,2} = \mathcal{F}(\Omega, \text{problem parameters, smoothness of } u)$$
Discrete Equations

Matrix Elements: $A_h^{\frac{1}{x}}$

$\varphi_i$ and $d\varphi_i/dx$...

\[ \begin{align*}
\varphi_i & \quad x_i \\
T_h^i & \quad T_h^{i+1} \\
0 & \quad 1
\end{align*} \]
Discrete Equations

Matrix Elements: $A_h^1$

...$\varphi_i$ and $d\varphi_i/dx$
Matrix Elements: $A^1_h$

Typical Row

$$A^1_{h \ i \ j} = \int_\Omega \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \ dx = \int_{T_h^i} \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \ dx + \int_{T_h^{i+1}} \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \ dx$$

is nonzero only for $i = j - 1, j, j + 1$

$$A^1_{h \ i \ i} = \frac{1}{h^2} (h) + \frac{1}{h^2} (h) = \frac{2}{h}$$

$$A^1_{h \ i \ i-1} = \frac{1}{h} (-\frac{1}{h}) (h) = -\frac{1}{h}$$

$$A^1_{h \ i \ i+1} = (-\frac{1}{h}) \frac{1}{h} (h) = -\frac{1}{h}$$
Matrix Elements: $A^1_h$

Discrete Equations

$$A^1_{h \ 1 \ 1} = \frac{2}{h}, \quad A^1_{h \ 1 \ 2} = -\frac{1}{h},$$

$$A^1_{h \ n \ n} = \frac{2}{h}, \quad A^1_{h \ n \ n-1} = -\frac{1}{h}.$$
**Discrete Equations**

Matrix Elements: $A_{h}^{1}$

Properties of $A_{h}$

$$A_{h}^{1} = \frac{1}{h} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & 0 & \ddots & \ddots & \\ & & 0 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$A_{h}^{1}$ is SPD; and diagonally dominant; and sparse; and tridiagonal.
$M_h \in \mathbb{R}^{n \times n}$:

$$M_{h\,i\,j} = \int_0^1 \varphi_i \varphi_j \, dx$$

the finite element “identity” ($I$) operator

Is nonzero only for $i = j - 1, j, j + 1$

$$M_{h\,i\,j} = \int_{T_h^i} \varphi_i \varphi_j \, dx + \int_{T_h^{i+1}} \varphi_i \varphi_j \, dx$$
For linear elements, nodal basis:

\[
M_h = h \begin{pmatrix}
\frac{2}{3} & \frac{1}{6} & 0 & \cdots & 0 \\
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \cdots & 0 \\
0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
0 & 0 & \frac{1}{6} & \frac{2}{3}
\end{pmatrix}
\]

sparse, banded, tri-diagonal — “close” to \( I \).
Discrete Equations

“Load” Vector Elements: $F_h$

\[ F_{h,i} = \int_0^1 f \varphi_i \, dx \]

\[ F_{h,i} = \int_{T_h^i} f \varphi_i \, dx + \int_{T_h^{i+1}} f \varphi_i \, dx, \quad i = 1, \ldots, n; \]
\( \underline{u}_h \in \mathbb{R}^n \) satisfies

\[
\begin{bmatrix}
A^1_h + \alpha M_h
\end{bmatrix}
\begin{pmatrix}
\underline{u}_h \\
\vdots \\
\underline{u}_n
\end{pmatrix}
=
\begin{pmatrix}
F_h \\
\vdots \\
F_n
\end{pmatrix}
\]
Non-dimensional form

\( k^i \): Thermal conductivity for \( \Omega_i, \ i = 0, \ldots, 4 \)

\( B_i \): Heat transfer coefficient

\( t \) : Geometric parameters

\( L \) : Geometric parameters
Finite element method

Example

\[ X_h = \text{span}\{\varphi_1, \ldots, \varphi_n\} \]

\(\varphi_i(x)\): Nodal basis functions

- First order elements

\[ \dim(X_h) = n \]
Example

Possible solutions

\[ \mu_1 \quad \mu_2 \quad \mu_3 \quad \mu_4 \]
Example

Extensions

- Complicated geometries
- General classes of problems
  (Good mathematical properties)
- Wider class of operators
Summary

• Why Poisson Equation
  – Reminder about heat conducting bar

• Finite-Difference And Basis function methods
  – Key question of convergence

• Convergence of Finite-Element methods
  – Key idea: solve Poisson by minimization
  – Demonstrate optimality in a carefully chosen norm