Problem 11.1 (OSB 11.3)

Note: The answers in the back of the book may not be correct in your version of the textbook.

We factor $|X(e^{j\omega})|^2$ into:

$$|X(e^{j\omega})|^2 = \frac{5}{4} - \cos \omega$$

$$= \left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{2}e^{j\omega}\right)$$

$$= X(e^{j\omega})X^\ast(e^{j\omega})$$

As a first attempt, we take

$$X(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}$$

$$x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

which does not satisfy the constraints $x[0] = 0$ and $x[1] > 0$.

We can modify the above choice by cascading it with an all-pass system, which will not affect the magnitude squared of the Fourier transform. Therefore we let

$$X(e^{j\omega}) = \left(1 - \frac{1}{2}e^{-j\omega}\right)e^{-j\omega}$$

$$x[n] = \delta[n-1] - \frac{1}{2}\delta[n-2]$$

which does satisfy all of the constraints.

Another choice that works is to take the second factor in $|X(e^{j\omega})|^2$ and cascade it with $(-e^{-j2\omega})$:

$$X(e^{j\omega}) = \left(1 - \frac{1}{2}e^{j\omega}\right)(-e^{-j2\omega}) = \frac{1}{2}e^{-j\omega} - e^{-j2\omega}$$

$$x[n] = \frac{1}{2}\delta[n-1] - \delta[n-2]$$

Note that this second choice uses the zero at $z = 2$, the conjugate reciprocal of the zero at $z = \frac{1}{2}$ in the first choice. Conjugate reciprocal zeroes yield the same Fourier transform magnitude (up to a scaling).
Problem 11.2
The inverse DTFT of \( j \text{Im}\{Y(e^{j\omega})\} \) is the odd part of \( y[n] \), denoted by \( y_o[n] \).

\[
y_o[n] = \text{DTFT}^{-1}[j3\sin \omega + j\sin 3\omega]
\]
\[
= \text{DTFT}^{-1}\left[\frac{1}{2} \left(3e^{j\omega} - 3e^{-j\omega} + e^{j3\omega} - e^{-j3\omega}\right)\right]
\]
\[
= \frac{1}{2} (3\delta[n + 1] - 3\delta[n - 1] + \delta[n + 3] - \delta[n - 3])
\]

Since \( y[n] \) is real and causal,

\[
y[n] = 2y_o[n]u[n] + y[0]\delta[n]
\]
\[
= y[0]\delta[n] - 3\delta[n - 1] - \delta[n - 3]
\]

To determine \( y[0] \), we use the fact that \( Y(e^{j\omega})\big|_{\omega=\pi} = 3 \), i.e.,

\[
Y(e^{j\omega})\big|_{\omega=\pi} = \sum_{n=-\infty}^{\infty} y[n](-1)^n
\]
\[
= y[0] + 3 + 1 = 3
\]
\[
y[0] = -1
\]

Therefore,

\[
y[n] = -\delta[n] - 3\delta[n - 1] - \delta[n - 3]
\]

Problem 11.3 (OSB 11.5)
In the frequency domain, the Hilbert transform is a 90° phase shifter:

\[
H(e^{j\omega}) = \begin{cases} 
-j, & 0 < \omega < \pi \\
 j, & -\pi < \omega < 0
\end{cases}
\]

To find the Hilbert transform of each sequence, we will take the Fourier transform, multiply by \( H(e^{j\omega}) \), and take the inverse Fourier transform.

(a)

\[
x_r[n] = \cos \omega_0 n
\]
\[
X_r(e^{j\omega}) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0), \quad -\pi < \omega \leq \pi
\]
\[
X_i(e^{j\omega}) = H(e^{j\omega})X_r(e^{j\omega})
\]
\[
= -j\pi \delta(\omega - \omega_0) + j\pi \delta(\omega + \omega_0), \quad -\pi < \omega \leq \pi
\]
\[
x_i[n] = \sin \omega_0 n
\]
(b) \[ x_r[n] = \sin \omega_0 n \]
\[ X_r(e^{j\omega}) = \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0), \quad -\pi < \omega \leq \pi \]
\[ X_i(e^{j\omega}) = -\pi \delta(\omega - \omega_0) - \pi \delta(\omega + \omega_0), \quad -\pi < \omega \leq \pi \]
\[ x_i[n] = -\cos \omega_0 n \]

(c) \( x_r[n] \) is the impulse response of an ideal low-pass filter with cut-off frequency \( \omega_c \):

\[ x_r[n] = \sin(\omega_c n) \]
\[ X_r(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \]
\[ X_i(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \omega_c \\ j, & -\omega_c < \omega < 0 \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \]
\[ x_i[n] = \frac{1}{2\pi} \int_{-\omega_c}^{0} j e^{j\omega n} d\omega - \frac{1}{2\pi} \int_{0}^{\omega_c} j e^{j\omega n} d\omega \]
\[ = 1 - \cos(\omega_c n) \]

**Problem 11.4**

The DTFT of \( y_1[n] \) is given by

\[ Y_1(e^{j\omega}) = X(e^{j\omega})e^{j\theta(\omega)}, \quad -\pi < \omega < \pi \]

The DTFT of \( y_2[n] \) is given by

\[ Y_2(e^{j\omega}) = \begin{cases} X(e^{j\omega})e^{j(\theta(\omega)-\pi/2)}, & 0 < \omega < \pi \\ X(e^{j\omega})e^{j(\theta(\omega)+\pi/2)}, & -\pi < \omega < 0 \end{cases} \]
\[ = \begin{cases} -jX(e^{j\omega})e^{j\theta(\omega)}, & 0 < \omega < \pi \\ jX(e^{j\omega})e^{j\theta(\omega)}, & -\pi < \omega < 0 \end{cases} \]
Since \( w[n] = y_1[n] + jy_2[n] \),

\[
W(e^{j\omega}) = Y_1(e^{j\omega}) + jY_2(e^{j\omega})
\]

\[
= \begin{cases} 
X(e^{j\omega})e^{j\theta(\omega)}(1 + 1), & 0 < \omega < \pi \\
X(e^{j\omega})e^{j\theta(\omega)}(1 - 1), & -\pi < \omega < 0 
\end{cases}
\]

\[
= \begin{cases} 
2X(e^{j\omega})e^{j\theta(\omega)}, & 0 < \omega < \pi \\
0, & -\pi < \omega < 0 
\end{cases}
\]

Therefore,

\[
W(e^{j\omega}) = 0, \quad -\pi < \omega < 0
\]

and since \(|e^{j\theta(\omega)}| = 1|\),

\[
|W(e^{j\omega})| = 2|X(e^{j\omega})|, \quad 0 < \omega < \pi
\]

**Problem 11.5**

We find the Fourier transform of \( h[n] \) and then take its complex logarithm,

\[
h[n] = \delta[n] + \alpha \delta[n - n_0]
\]

\[
H(e^{j\omega}) = 1 + \alpha e^{-j\omega n_0}
\]

\[
\hat{H}(e^{j\omega}) = \log (1 + \alpha e^{-j\omega n_0})
\]

The power series expansion for \( \log(1 + x) \) with \(|x| < 1\) is given by:

\[
\log(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}
\]

Letting \( x = \alpha e^{-j\omega n_0} \) and checking that \(|x| = |\alpha e^{-j\omega n_0}| = |\alpha| < 1\) as assumed, we obtain:

\[
\hat{H}(e^{j\omega}) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\alpha^k}{k} e^{-j\omega kn_0}
\]

The complex cepstrum \( \hat{h}[n] \) is found by taking the inverse Fourier transform of \( \hat{H}(e^{j\omega}) \) and identifying \( e^{-j\omega kn_0} \leftrightarrow \delta[n - kn_0] \):

\[
\hat{h}[n] = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\alpha^k}{k} \delta[n - kn_0]
\]

\( \hat{h}[n] \) is plotted in Figure 11.5-1:
Problem 11.6

The sequence \( x[n] \) being minimum-phase means that \( x[n] \) is also causal, so that \( x[n] = 0, \ n < 0 \). As stated in the problem, minimum phase implies that the complex cepstrum \( \hat{x}[n] \) is causal, i.e. \( \hat{x}[n] = 0, \ n < 0 \). Thus the lower bound on the sum in equation (12.34) becomes \( k = 0 \) (because of \( \hat{x}[k] \)), while the upper bound becomes \( k = n \) (because of \( x[n-k] \)).

\[
  x[n] = \sum_{k=0}^{n} \left( \frac{k}{n} \right) \hat{x}[k] x[n-k], \quad n > 0
\]

Isolating the \( k = n \) term from the sum and solving for \( \hat{x}[n] \),

\[
  x[n] = \hat{x}[n] x[0] + \sum_{k=0}^{n-1} \left( \frac{k}{n} \right) \hat{x}[k] x[n-k]
\]

\[
  \hat{x}[n] = \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \left( \frac{k}{n} \right) \hat{x}[k] \frac{x[n-k]}{x[0]}, \quad n > 0
\]

The equation above is a recursion formula for \( \hat{x}[n], \ n > 0 \), while we know that \( \hat{x}[n] = 0 \) for \( n < 0 \):

\[
  \hat{x}[n] = \begin{cases} 0, & n < 0 \\ \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \left( \frac{k}{n} \right) \hat{x}[k] \frac{x[n-k]}{x[0]}, & n > 0 \end{cases}
\]

However, the recursion cannot determine \( \hat{x}[0] \), so we must find it through some other means. Recall the initial-value theorem for a causal sequence \( x[n] \) such that \( x[n] = 0 \) for \( n < 0 \):

\[
  x[0] = \lim_{z \to \infty} X(z)
\]
Since \( \hat{x}[n] \) is also zero for \( n < 0 \),
\[
\hat{x}[0] = \lim_{z \to \infty} \hat{X}(z)
\]
But \( \hat{X}(z) = \log X(z) \), so we have
\[
\hat{x}[0] = \lim_{z \to \infty} \log X(z)
= \log \left( \lim_{z \to \infty} X(z) \right)
= \log (x[0])
\]
Thus we can determine \( \hat{x}[0] \) using only \( x[0] \), so the computation is causal.

Now suppose that \( \hat{x}[n] \) is known for \( 0 \leq n \leq n_0 - 1 \). Using the recursion, we are able to calculate the next value \( \hat{x}[n_0] \) from the known past values of \( \hat{x}[n] \) and from the values of \( x[n] \) for \( 0 \leq n \leq n_0 \). To start the recursion, we determine \( \hat{x}[0] \), which is then used to determine \( \hat{x}[1] \), and so on. \( \hat{x}[n] \) can therefore be recursively computed. Furthermore, the computation of \( \hat{x}[n_0] \) for any \( n_0 \geq 0 \) only involves values of \( x[n] \) for \( 0 \leq n \leq n_0 \), so the recursion can be implemented in a causal manner.

**Problem 11.7**

(a) Similar to Problem 11.2, the inverse DTFT of \( \text{Re}\{X(e^{j\omega})\} \) is the even part \( x_e[n] \) of \( x[n] \).
\[
x_e[n] = \text{DTFT}^{-1}[1 + 3 \cos \omega + \cos 3\omega] = \delta[n] + \frac{1}{2} (3\delta[n + 1] + 3\delta[n - 1] + \delta[n + 3] + \delta[n - 3])
\]
Since \( x[n] \) is real and causal, it can be uniquely determined from its even part \( x_e[n] \):
\[
x[n] = 2x_e[n]u[n] - x_e[0]\delta[n]
= \delta[n] + 3\delta[n - 1] + \delta[n - 3]
\]

(b) Let \( X(e^{j\omega}) \) and \( \hat{X}(e^{j\omega}) \) denote the Fourier transforms of the sequence \( x[n] \) and its complex cepstrum \( \hat{x}[n] \). \( X(e^{j\omega}) \) and \( \hat{X}(e^{j\omega}) \) are related by:
\[
\hat{X}(e^{j\omega}) = \log |X(e^{j\omega})| + j \text{arg} \left[ X(e^{j\omega}) \right]
\]
where \( \text{arg} \left[ X(e^{j\omega}) \right] \) denotes the continuous unwrapped phase.
If \( x_1[n] = x[-n] \), then \( X_1(e^{j\omega}) = X(e^{-j\omega}) \), and
\[
\hat{X}_1(e^{j\omega}) = \log |X_1(e^{j\omega})| + j \text{arg} \left[ X_1(e^{j\omega}) \right]
= \log |X(e^{-j\omega})| + j \text{arg} \left[ X(e^{-j\omega}) \right]
= \hat{X}(e^{-j\omega})
\]
Therefore $\hat{x}_1[n] = \hat{x}[-n]$ also. Statement 1 is true.

If $x[n]$ is real, then the Fourier transform magnitude $|X(e^{j\omega})|$ is an even function of $\omega$, while the unwrapped phase is an odd function of $\omega$. The real part of $\hat{X}(e^{j\omega})$, which is the logarithm of $|X(e^{j\omega})|$, must be an even function, while the imaginary part of $\hat{X}(e^{j\omega})$ is equal to $\arg X(e^{j\omega})$ and must be an odd function. Therefore $\hat{X}(e^{j\omega})$ is conjugate symmetric and the complex cepstrum $\hat{x}[n]$ is real.

Statement 2 is also true.