Measure and complements

We listed the rational numbers in $[-T/2, T/2]$ as $a_1, a_2, \ldots$

$$\mu\left\{ \bigcup_{i=1}^{k} a_i \right\} = \sum_{i=1}^{k} \mu([a_i, a_i]) = 0$$

The complement of $\bigcup_{i=1}^{k} a_i$ is $\bigcap_{i=1}^{k} \overline{a_i}$ where $\overline{a_i}$ is all $t \in [-T/2, T/2]$ except $a_i$.

Thus $\bigcap_{i=1}^{k} \overline{a_i}$ is a union of $k+1$ intervals, filling $[-T/2, T/2]$ except $a_1, \ldots, a_k$.

In the limit, this is the union of an uncountable set of irrational numbers; the measure is $T$. 
MEASURABLE FUNCTIONS

A function \( \{u(t) : \mathbb{R} \rightarrow \mathbb{R}\} \) is measurable if \( \{t : u(t) < b\} \) is measurable for each \( b \in \mathbb{R} \).

The Lebesgue integral exists if the function is measurable and if the limit in the figure exists.

Horizontal crosshatching is what is added when \( \varepsilon \rightarrow \varepsilon/2 \). For \( u(t) \geq 0 \), the integral must exist (with perhaps an infinite value).
For $u(t) \geq 0$, the Lebesgue approximation might be infinite for all $\varepsilon$. Example: $u(t) = |1/t|$.

If approximation finite for any $\varepsilon$, then changing $\varepsilon$ to $\varepsilon/2$ adds at most $\varepsilon/2$ to approximation.

Continued halving of interval adds at most $\varepsilon/2 + \varepsilon/4 + \cdots + \varepsilon$.

If any approximation is finite, integral is finite.
For a positive and negative function \( u(t) \) define a positive and negative part:

\[
\begin{align*}
    u^+(t) &= \begin{cases} 
    u(t) & \text{for } t : u(t) \geq 0 \\
    0 & \text{for } t : u(t) < 0
    \end{cases} \\
    u^-(t) &= \begin{cases} 
    0 & \text{for } t : u(t) \geq 0 \\
    -u(t) & \text{for } t : u(t) < 0
    \end{cases}
\end{align*}
\]

\[ u(t) = u^+(t) - u^-(t). \]

If \( u(t) \) is measurable, then \( u^+(t) \) and \( u^-(t) \) are also and can be integrated as before.

\[
\int u(t) = \int u^+(t) - \int u^-(t) \, dt.
\]

except if both \( \int u^+(t) \, dt \) and \( \int u^-(t) \, dt \) are infinite, then the integral is undefined.
For \( \{ u(t) : [-T/2, T/2] \rightarrow \mathbb{R} \} \), the functions \(|u(t)|\) and \(|u(t)|^2\) are non-negative.

They are measurable if \( u(t) \) is.

\[ |u(t)| = u^+(t) + u^-(t) \quad \text{thus} \quad \int |u(t)| \, dt = \int u^+(t) \, dt + \int u^-(t) \, dt \]

**Def:** \( u(t) \) is \( \mathcal{L}_1 \) if measurable and \( \int |u(t)| \, dt < \infty \).

**Def:** \( u(t) \) is \( \mathcal{L}_2 \) if measurable and \( \int |u(t)|^2 \, dt < \infty \).
A complex function \( \{u(t) : [-T/2, T/2] \to \mathbb{C}\} \) is measurable if both \( \Re[u(t)] \) and \( \Im[u(t)] \) are measurable.

**Def:** \( u(t) \) is \( \mathcal{L}_1 \) if \( \int |u(t)| \, dt < \infty \).

Since \( |u(t)| \leq |\Re(u(t))| + |\Im(u(t))| \), it follows that \( u(t) \) is \( \mathcal{L}_1 \) if and only if \( \Re[u(t)] \) and \( \Im[u(t)] \) are \( \mathcal{L}_1 \).

**Def:** \( u(t) \) is \( \mathcal{L}_2 \) if \( \int |u(t)|^2 \, dt < \infty \). This happens if and only if \( \Re[u(t)] \) and \( \Im[u(t)] \) are \( \mathcal{L}_2 \).
If \(|u(t)| \geq 1\) for given \(t\), then \(|u(t)| \leq |u(t)|^2\).

Otherwise \(|u(t)| \leq 1\). For all \(t\),
\[
|u(t)| \leq |u(t)|^2 + 1.
\]

For \(\{u(t) : [-T/2, T/2 \to \mathbb{C}]\}\),
\[
\int_{-T/2}^{T/2} |u(t)| \, dt \leq \int_{-T/2}^{T/2} [|u(t)|^2 + 1] \, dt
\]
\[
= T + \int_{-T/2}^{T/2} |u(t)|^2 \, dt
\]

Thus \(L_2\) finite duration functions are also \(L_1\).
\( L_2 \text{ functions } [-T/2, T/2] \to \mathbb{C} \)

\( L_1 \text{ functions } [-T/2, T/2] \to \mathbb{C} \)

Measurable functions \([-T/2, T/2] \to \mathbb{C}\)
Back to Fourier series:

Note that $|u(t)| = |u(t)e^{2\pi ift}|$

Thus, if $\{u(t) : [−T/2, T/2] \rightarrow \mathbb{C}\}$ is $L_1$, then

$$\int |u(t)e^{2\pi ift}| dt < \infty.$$

$$|\int u(t)e^{2\pi ift} dt| \leq \int |u(t)| dt < \infty.$$

If $u(t)$ is $L_2$ and time-limited, it is $L_1$ and same conclusion follows.
**Theorem:** Let \( \{u(t) : [-T/2, T/2] \to \mathbb{C}\} \) be an \( \mathcal{L}_2 \) function. Then for each \( k \in \mathbb{Z} \), the Lebesgue integral

\[
\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi ikt/T} \, dt
\]

exists and satisfies \( |\hat{u}_k| \leq \frac{1}{T} \int |u(t)| \, dt < \infty \). Furthermore,

\[
\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \sum_{k=-k_0}^{k_0} \hat{u}_k e^{2\pi ikt/T} \right|^2 \, dt = 0,
\]

where the limit is monotonic in \( k_0 \).
The most important part of the theorem is that

\[ u(t) \approx \sum_{k=-k_0}^{k_0} \hat{u}_k e^{2\pi i k t/T} \]

where the energy difference between the terms goes to 0 as \( k_0 \to \infty \), i.e.,

\[
\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \sum_{k=-k_0}^{k_0} \hat{u}_k e^{2\pi i k t/T} \right|^2 \, dt = 0,
\]

We abbreviate this convergence by

\[ u(t) = \text{l.i.m.} \sum_{k} \hat{u}_k e^{2\pi i k t/T} \text{rect}(\frac{t}{T}). \]
\[ u(t) = \text{l.i.m.} \sum_k \hat{u}_k e^{2\pi i k t / T} \text{rect}(\frac{t}{T}). \]

This does not mean that the sum on the right converges to \( u(t) \) at each \( t \) and does not mean that the sum converges to anything.

There is an important theorem by Carleson that says that for \( L_2 \) functions, the sum converges a.e. That is, it converges to \( u(t) \) except on a set of \( t \) of measure 0.

This means that it converges for all integration purposes.
It is often important to go from sequence to function. The relevant result about Fourier series then is

**Theorem:** If a sequence of complex numbers \( \{\hat{u}_k; k \in \mathbb{Z}\} \) satisfies \( \sum_k |\hat{u}_k|^2 \), then an \( L_2 \) function \( \{u(t) : [-T/2, T/2] \to \mathbb{C}\} \) exists satisfying

\[
    u(t) = \text{l.i.m.} \sum_k \hat{u}_k e^{2\pi ikt/T} \text{rect}(\frac{t}{T}).
\]
Aside from all the mathematical hoopla (which is important), there is a very simple reason why so many things are simple with Fourier series. The expansion functions,

\[ \theta_k(t) = e^{2\pi i k t / T} \text{rect}(t/T) \]

are orthogonal. That is

\[ \int \theta_k(t)\theta_j^*(t) \, dt = T\delta_{k,j} \]

This is the feature that let us solve for \( \hat{u}_k(t) \) from the Fourier series \( u(t) = \sum_k \hat{u}_k \theta_k(t) \).
Functions not limited in time

We can segment an arbitrary $L_2$ function into segments of width $T$. The $m$th segment is $u_m(t) = u(t)\text{rect}(t/T - m)$. We then have

$$u(t) = \lim_{m_0 \to \infty} \sum_{m=-m_0}^{m_0} u_m(t)$$

This works because $u(t)$ is $L_2$. The energy in $u_m(t)$ must go to 0 as $m \to \infty$.

By shifting $u_m(t)$, we get the Fourier series:

$$u_m(t) = \lim_{m_0 \to \infty} \sum_k \hat{u}_{k,m} e^{2\pi i k t/T} \text{rect}(t/T - m), \quad \text{where}$$

$$\hat{u}_{k,m} = \frac{1}{T} \int_{-\infty}^{\infty} u(t) e^{-2\pi i k t/T} \text{rect}(t/T - m) \, dt, \quad -\infty < k < \infty.$$
This breaks $u(t)$ into a double sum expansion of orthogonal functions, first over segments, then over frequencies.

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m} e^{2\pi i k t/T} \text{rect}(\frac{t}{T} - m)$$

This is the first of a number of orthogonal expansions of arbitrary $L_2$ functions.

We call this the $T$-spaced truncated sinusoid expansion.
\[ u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m} e^{2\pi i k t/T} \text{rect} \left( \frac{t}{T} - m \right) \]

This is the conceptual basis for algorithms such as voice compression that segment the waveform and then process each segment.

It matches our intuition about frequency well; that is, in music, notes (frequencies) keep changing.

The awkward thing is that the segmentation parameter \( T \) is arbitrary and not fundamental.
Fourier transform: \( u(t) : \mathbb{R} \rightarrow \mathbb{C} \) to \( \hat{u}(f) : \mathbb{R} \rightarrow \mathbb{C} \)

\[
\hat{u}(f) = \int_{-\infty}^{\infty} u(t) e^{-2\pi i ft} \, dt.
\]

\[
u(t) = \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i ft} \, df.
\]

For “well-behaved functions,” first integral exists for all \( f \), second exists for all \( t \) and results in original \( u(t) \).

What does well-behaved mean? It means that the above is true.
\[ au(t) + bv(t) \leftrightarrow a\hat{u}(f) + b\hat{v}(f). \]
\[ u^*(-t) \leftrightarrow \hat{u}^*(f). \]
\[ \hat{u}(t) \leftrightarrow u(-f). \]
\[ u(t - \tau) \leftrightarrow e^{-2\pi if\tau}\hat{u}(f) \]
\[ u(t)e^{2\pi if_0t} \leftrightarrow \hat{u}(f - f_0) \]
\[ u(t/T) \leftrightarrow T\hat{u}(fT). \]
\[ \frac{du(t)}{dt} \leftrightarrow i2\pi f\hat{u}(f). \]
\[ \int_{-\infty}^{\infty} u(\tau)v(t-\tau)\,d\tau \leftrightarrow \hat{u}(f)\hat{v}(f). \]
\[ \int_{-\infty}^{\infty} u(\tau)v^*(\tau-t)\,d\tau \leftrightarrow \hat{u}(f)\hat{v}^*(f). \]
Two useful special cases of any Fourier transform pair are:

\[
    u(0) = \int_{-\infty}^{\infty} \hat{u}(f) \, df;
\]

\[
    \hat{u}(0) = \int_{-\infty}^{\infty} u(t) \, dt.
\]

Parseval’s theorem:

\[
    \int_{-\infty}^{\infty} u(t) v^*(t) \, dt = \int_{-\infty}^{\infty} \hat{u}(f) \hat{v}^*(f) \, df.
\]

Replacing \( v(t) \) by \( u(t) \) yields the energy equation,

\[
    \int_{-\infty}^{\infty} |u(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2 \, df.
\]
6.450 Principles of Digital Communication I
Fall 2009

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.