Chapter 4

Source and channel waveforms

4.1 Introduction

This chapter has a dual objective. The first is to understand *analog data compression*, *i.e.*, the compression of sources such as voice for which the output is an arbitrarily varying real or complex valued function of time; we denote such functions as *waveforms*. The second is to begin studying the waveforms that are typically transmitted at the input and received at the output of communication channels. The same set of mathematical tools are needed for the understanding and representation of both source and channel waveforms; the development of these results is the central topic in this chapter.

These results about waveforms are standard topics in mathematical courses on analysis, real and complex variables, functional analysis, and linear algebra. They are stated here without the precision or generality of a good mathematics text, but with considerably more precision and interpretation than is found in most engineering texts.

4.1.1 Analog sources

The output of many analog sources (voice is the typical example) can be represented as a waveform,[1] $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ or $\{u(t) : \mathbb{R} \to \mathbb{C}\}$. Often, as with voice, we are interested only in real waveforms, but the simple generalization to complex waveforms is essential for Fourier analysis and for baseband modeling of communication channels. Since a real valued function can be viewed as a special case of a complex valued function, the results for complex functions are also useful for real functions.

We observed earlier that more complicated analog sources such as video can be viewed as mappings from $\mathbb{R}^n$ to $\mathbb{R}$, *e.g.*, as mappings from horizontal/vertical position and time to real analog values, but for simplicity we consider only waveform sources here.

Recall why it is desirable to convert analog sources into bits:

- The use of a standard binary interface separates the problem of compressing sources from

[1] The notation $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ refers to a function that maps each real number $t \in \mathbb{R}$ into another real number $u(t) \in \mathbb{R}$. Similarly, $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ maps each real number $t \in \mathbb{R}$ into a complex number $u(t) \in \mathbb{C}$. These functions of time, *i.e.*, these waveforms, are usually viewed as dimensionless, thus allowing us to separate physical scale factors in communication problems from the waveform shape.

the problems of channel coding and modulation.

- The outputs from multiple sources can be easily multiplexed together. Multiplexers can work by interleaving bits, 8-bit bytes, or longer packets from different sources.
- When a bit sequence travels serially through multiple links (as in a network), the noisy bit sequence can be cleaned up (regenerated) at each intermediate node, whereas noise tends to gradually accumulate with noisy analog transmission.

A common way of encoding a waveform into a bit sequence is as follows:

1. Approximate the analog waveform $\{u(t); t \in \mathbb{R}\}$ by its samples[2] $\{u(mT); m \in \mathbb{Z}\}$ at regularly spaced sample times, $\dots, -T, 0, T, 2T, \dots$.
2. Quantize each sample (or n-tuple of samples) into a quantization region.
3. Encode each quantization region (or block of regions) into a string of bits.

These three layers of encoding are illustrated in Figure 4.1, with the three corresponding layers of decoding.

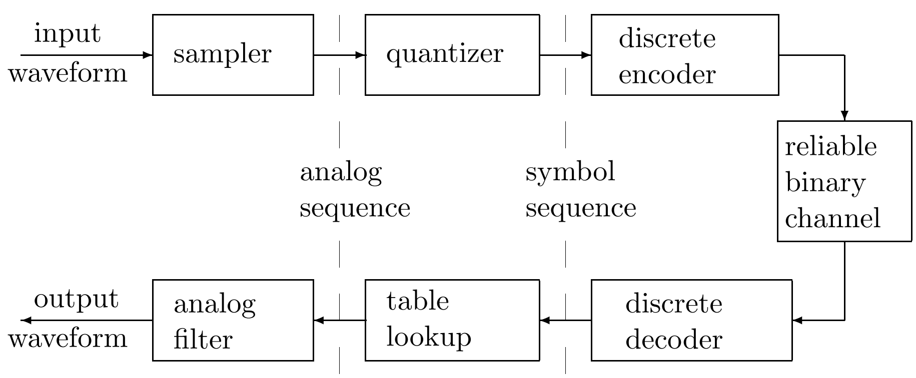


Figure 4.1: Encoding and decoding a waveform source.

Example 4.1.1. In standard telephony, the voice is filtered to 4000 Hertz (4 kHz) and then sampled at 8000 samples per second.[3] Each sample is then quantized to one of 256 possible levels, represented by 8 bits. Thus the voice signal is represented as a 64 kilobit/second (kb/s) sequence. (Modern digital wireless systems use more sophisticated voice coding schemes that reduce the data rate to about 8 kb/s with little loss of voice quality.)

The sampling above may be generalized in a variety of ways for converting waveforms into sequences of real or complex numbers. For example, modern voice compression techniques first

[2] $\mathbb{Z}$ denotes the set of integers, $-\infty < m < \infty$, so $\{u(mT); m \in \mathbb{Z}\}$ denotes the doubly infinite sequence of samples with $-\infty < m < \infty$

[3] The sampling theorem, to be discussed in Section 4.6, essentially says that if a waveform is baseband-limited to W Hz, then it can be represented perfectly by 2W samples per second. The highest note on a piano is about 4 kHz, which is considerably higher than most voice frequencies.

segment the voice waveform into 20 msec. segments and then use the frequency structure of each segment to generate a vector of numbers. The resulting vector can then be quantized and encoded as discussed before.

An individual waveform from an analog source should be viewed as a sample waveform from a *random process*. The resulting probabilistic structure on these sample waveforms then determines a probability assignment on the sequences representing these sample waveforms. This random characterization will be studied in Chapter 7; for now, the focus is on ways to map deterministic waveforms to sequences and *vice versa*. These mappings are crucial both for source coding and channel transmission.

4.1.2 Communication channels

Some examples of communication channels are as follows: a pair of antennas separated by open space; a laser and an optical receiver separated by an optical fiber; and a microwave transmitter and receiver separated by a wave guide. For the antenna example, a real waveform at the input in the appropriate frequency band is converted by the input antenna into electromagnetic radiation, part of which is received at the receiving antenna and converted back to a waveform. For many purposes, these physical channels can be viewed as black boxes where the output waveform can be described as a function of the input waveform and noise of various kinds.

Viewing these channels as black boxes is another example of layering. The optical or microwave devices or antennas can be considered as an inner layer around the actual physical channel. This layered view will be adopted here for the most part, since the physics of antennas, optics, and microwave are largely separable from the digital communication issues developed here. One exception to this is the description of physical channels for wireless communication in Chapter 9. As will be seen, describing a wireless channel as a black box requires some understanding of the underlying physical phenomena.

The function of a channel encoder, *i.e.*, a modulator, is to convert the incoming sequence of binary digits into a waveform in such a way that the noise corrupted waveform at the receiver can, with high probability, be converted back into the original binary digits. This is typically done by first converting the binary sequence into a sequence of analog signals, which are then converted to a waveform. This procession - bit sequence to analog sequence to waveform - is the same procession as performed by a source decoder, and the opposite to that performed by the source encoder. How these functions should be accomplished is very different in the source and channel cases, but both involve converting between waveforms and analog sequences.

The waveforms of interest for channel transmission and reception should be viewed as sample waveforms of random processes (in the same way that source waveforms should be viewed as sample waveforms from a random process). This chapter, however, is concerned only about the relationship between deterministic waveforms and analog sequences; the necessary results about random processes will be postponed until Chapter 7. The reason why so much mathematical precision is necessary here, however, is that these waveforms are a priori unknown. In other words, one cannot use the conventional engineering approach of performing some computation on a function and assuming it is correct if an answer emerges[4].

[4]This is not to disparage the use of computational (either hand or computer) techniques to get a quick answer without worrying about fine points. Such techniques often provides insight and understanding, and the fine points can be addressed later. For a random process, however, one doesn't know a priori which sample functions can provide computational insight.

4.2 Fourier series

Perhaps the simplest example of an analog sequence that can represent a waveform comes from the Fourier series. The Fourier series is also useful in understanding Fourier transforms and discrete-time Fourier transforms (DTFTs). As will be explained later, our study of these topics will be limited to finite-energy waveforms. Useful models for source and channel waveforms almost invariably fall into the finite-energy class.

The Fourier series represents a waveform, either periodic or time-limited, as a weighted sum of sinusoids. Each weight (coefficient) in the sum is determined by the function, and the function is essentially determined by the sequence of weights. Thus the function and the sequence of weights are essentially equivalent representations.

Our interest here is almost exclusively in time-limited rather than periodic waveforms[5]. Initially the waveforms are assumed to be time-limited to some interval $-T/2 \le t \le T/2$ of an arbitrary duration $T > 0$ around 0. This is then generalized to time-limited waveforms centered at some arbitrary time. Finally, an arbitrary waveform is segmented into equal length segments each of duration T; each such segment is then represented by a Fourier series. This is closely related to modern voice-compression techniques where voice waveforms are segmented into 20 msec intervals, each of which are separately expanded into a Fourier-like series.

Consider a complex function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ that is nonzero only for $-T/2 \le t \le T/2$ (*i.e.*, $u(t) = 0$ for $t < -T/2$ and $t > T/2$). Such a function is frequently indicated by $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$. The *Fourier series* for such a time-limited function is given by[6]

$$u(t) = \begin{cases} \sum_{k=-\infty}^{\infty} \hat{u}_k \, e^{2\pi ikt/T} & \text{for } -T/2 \le t \le T/2 \\ 0 & \text{elsewhere,} \end{cases} \tag{4.1}$$

where i denotes[7] $\sqrt{-1}$. The Fourier series coefficients $\hat{u}_k$ are in general complex (even if $u(t)$ is real), and are given by

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi ikt/T} \, dt, \qquad -\infty < k < \infty. \tag{4.2}$$

The standard rectangular function,

$$\text{rect}(t) = \begin{cases} 1 & \text{for } -1/2 \le t \le 1/2 \\ 0 & \text{elsewhere,} \end{cases}$$

can be used to simplify (4.1) as follows:

$$u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k \, e^{2\pi ikt/T} \, \text{rect}(\frac{t}{T}). \tag{4.3}$$

This expresses $u(t)$ as a linear combination of truncated complex sinusoids,

$$u(t) = \sum_{k \in \mathbb{Z}} \hat{u}_k \theta_k(t) \qquad \text{where} \quad \theta_k(t) = e^{2\pi ikt/T} \, \text{rect}(\frac{t}{T}). \tag{4.4}$$

[5]Periodic waveforms are not very interesting for carrying information; after observing one period, the rest of the waveform carries nothing new.

[6]The conditions and the sense in which (4.1) holds are discussed later.

[7]The use of i for $\sqrt{-1}$ is standard in all scientific fields except electrical engineering. Electrical engineers formerly reserved the symbol i for electrical current and thus often use j to denote $\sqrt{-1}$.

Assuming that (4.4) holds for some set of coefficients $\{\hat{u}_k;\, k \in \mathbb{Z}\}$, the following simple and instructive argument shows why (4.2) is satisifed for that set of coefficients. Two complex waveforms, $\theta_k(t)$ and $\theta_m(t)$, are defined to be *orthogonal* if $\int_{-\infty}^{\infty} \theta_k(t)\theta_m^*(t)\,dt = 0$. The truncated complex sinusoids in (4.4) are orthogonal since the interval $[-T/2, T/2]$ contains an integral number of cycles of each, *i.e.*, for $k \neq m \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} \theta_k(t)\theta_m^*(t)\,dt \;=\; \int_{-T/2}^{T/2} e^{2\pi i(k-m)t/T}\,dt \;=\; 0.$$

Thus the right side of (4.2) can be evaluated as

$$\begin{aligned}
\frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-2\pi ikt/T}\,dt &= \frac{1}{T}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \hat{u}_m\theta_m(t)\theta_k^*(t)\,dt \\
&= \frac{\hat{u}_k}{T}\int_{-\infty}^{\infty} |\theta_k(t)|^2\,dt \\
&= \frac{\hat{u}_k}{T}\int_{-T/2}^{T/2} dt \;=\; \hat{u}_k.
\end{aligned} \tag{4.5}$$

An expansion such as that of (4.4) is called an *orthogonal expansion*. As shown later, the argument in (4.5) can be used to find the coefficients in any orthogonal expansion. At that point, more care will be taken in exchanging the order of integration and summation above.

Example 4.2.1. This and the following example illustrate why (4.4) need not be valid for all values of t. Let $u(t) = \text{rect}(2t)$ (see Figure 4.2). Consider representing $u(t)$ by a Fourier series over the interval $-1/2 \le t \le 1/2$. As illustrated, the series can be shown to converge to $u(t)$ at all $t \in [-1/2, 1/2]$ except for the discontinuities at $t = \pm 1/4$. At $t = \pm 1/4$, the series converges to the midpoint of the discontinuity and (4.4) is not valid[8] at $t = \pm 1/4$. The next section will show how to state (4.4) precisely so as to avoid these convergence issues.

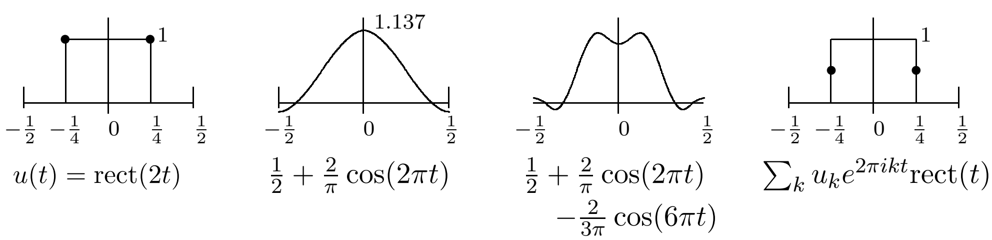
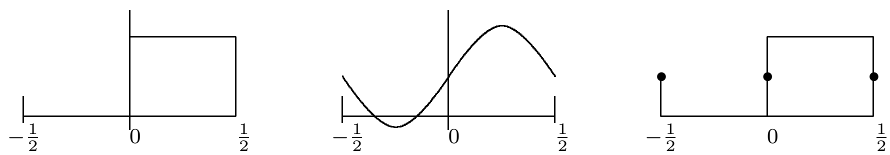


Figure 4.2: The Fourier series (over $[-1/2,\ 1/2]$) of a rectangular pulse. The second figure depicts a partial sum with $k = -1, 0, 1$ and the third figure depicts a partial sum with $-3 \le k \le 3$. The right figure illustrates that the series converges to $u(t)$ except at the points $t = \pm 1/4$, where it converges to 1/2.

Example 4.2.2. As a variation of the previous example, let $v(t)$ be 1 for $0 \le t \le 1/2$ and 0 elsewhere. Figure 4.3 shows the corresponding Fourier series over the interval $-1/2 \le t \le 1/2$.

[8]Most engineers, including the author, would say 'so what, who cares what the Fourier series converges to at a discontinuity of the waveform'. Unfortunately, this example is only the tip of an iceberg, especially when time-sampling of waveforms and sample waveforms of random processes are considered.

A peculiar feature of this example is the isolated discontinuity at $t = -1/2$, where the series converges to $1/2$. This happens because the untruncated Fourier series, $\sum_{k=-\infty}^{\infty} \hat{v}_k e^{2\pi ikt}$, is periodic with period 1 and thus must have the same value at both $t = -1/2$ and $t = 1/2$. More generally, if an arbitrary function $\{v(t) : [-T/2, T/2] \to \mathbb{C}\}$ has $v(-T/2) \neq v(T/2)$, then its Fourier series over that interval cannot converge to $v(t)$ at both those points.



$v(t) = \text{rect}(2t - \frac{1}{4})$ $\qquad$ $\frac{1}{2} + \frac{2}{\pi}\sin(2\pi t)$ $\qquad$ $\sum_{k=-\infty}^{\infty} v_k e^{2\pi ikt}\text{rect}(t)$

Figure 4.3: The Fourier series over $[-1/2, 1/2]$ of the same rectangular pulse shifted right by $1/4$. The middle figure again depicts a partial expansion with $k = -1, 0, 1$. The right figure shows that the series converges to $v(t)$ except at the points $t = -1/2, 0$, and $1/2$, at each of which it converges to $1/2$.

4.2.1 Finite-energy waveforms

The *energy* in a real or complex waveform $u(t)$ is defined[9] to be $\int_{-\infty}^{\infty} |u(t)|^2 \, dt$. The energy in source waveforms plays a major role in determining how well the waveforms can be compressed for a given level of distortion. As a preliminary explanation, consider the energy in a time-limited waveform $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$. This energy is related to the Fourier series coefficients of $u(t)$ by the following *energy equation* which is derived in Exercise 4.2 by the same argument used in (4.5):

$$\int_{t=-T/2}^{T/2} |u(t)|^2 \, dt = T \sum_{k=-\infty}^{\infty} |\hat{u}_k|^2. \tag{4.6}$$

Suppose that $u(t)$ is compressed by first generating its Fourier series coefficients, $\{\hat{u}_k;\ k \in \mathbb{Z}\}$ and then compressing those coefficients. Let $\{\hat{v}_k; k \in \mathbb{Z}\}$ be this sequence of compressed coefficients. Using a squared distortion measure for the coefficients, the overall distortion is $\sum_k |\hat{u}_k - \hat{v}_k|^2$. Suppose these compressed coefficients are now encoded, sent through a channel, reliably decoded, and converted back to a waveform $v(t) = \sum_k \hat{v}_k e^{2\pi ikt/T}$ as in Figure 4.1. The difference between the input waveform $u(t)$ and the output $v(t)$ is then $u(t) - v(t)$, which has the Fourier series $\sum_k (\hat{u}_k - \hat{v}_k) e^{2\pi ikt/T}$. Substituting $u(t) - v(t)$ into (4.6) results in the *difference-energy equation*,

$$\int_{t=-T/2}^{T/2} |u(t) - v(t)|^2 \, dt = T \sum_k |\hat{u}_k - \hat{v}_k|^2. \tag{4.7}$$

Thus the energy in the difference between $u(t)$ and its reconstruction $v(t)$ is simply T times the sum of the squared differences of the quantized coefficients. This means that reducing the squared difference in the quantization of a coefficient leads directly to reducing the energy in the waveform difference. The energy in the waveform difference is a common and reasonable

[9]Note that $u^2 = |u|^2$ if u is real, but for complex u, u^2 can be negative or complex and $|u|^2 = uu^* = [\Re(u)]^2 + [\Im(u)]^2$ is required to correspond to the intuitive notion of energy.

measure of distortion, but the fact that it is directly related to mean-squared coefficient distortion provides an important added reason for its widespread use.

There must be at least T units of delay involved in finding the Fourier coefficients for $u(t)$ in $[-T/2, T/2]$ and then reconstituting $v(t)$ from the quantized coefficients at the receiver. There is additional processing and propagation delay in the channel. Thus the output waveform must be a delayed approximation to the input. All of this delay is accounted for by timing recovery processes at the receiver. This timing delay is set so that $v(t)$ at the receiver, according to the receiver timing, is the appropriate approximation to $u(t)$ at the transmitter, according to the transmitter timing. Timing recovery and delay are important problems, but they are largely separable from the problems of current interest. Thus, after recognizing that receiver timing is delayed from transmitter timing, delay can be otherwise ignored for now.

Next, visualize the Fourier coefficients $\hat{u}_k$ as sample values of independent random variables and visualize $u(t)$, as given by (4.3), as a sample value of the corresponding random process (this will be explained carefully in Chapter 7). The expected energy in this random process is equal to T times the sum of the mean-squared values of the coefficients. Similarly the expected energy in the difference between $u(t)$ and $v(t)$ is equal to T times the sum of the mean-squared coefficient distortions. It was seen by scaling in Chapter 3 that the the mean-squared quantization error for an analog random variable is proportional to the variance of that random variable. It is thus not surprising that the expected energy in a random waveform will have a similar relation to the mean-squared distortion after compression.

There is an obvious practical problem with compressing a finite-duration waveform by quantizing an *infinite* set of coefficients. One solution is equally obvious: compress only those coefficients with a significant mean-squared value. Since the expected value of $\sum_k |\hat{u}_k|^2$ is finite for finite-energy functions, the mean-squared distortion from ignoring small coefficients can be made as small as desired by choosing a sufficiently large finite set of coefficients. One then simply chooses $\hat{v}_k = 0$ in (4.7) for each ignored value of k.

The above argument will be developed carefully after developing the required tools. For now, there are two important insights. First, the energy in a source waveform is an important parameter in data compression, and second, the source waveforms of interest will have finite energy and can be compressed by compressing a finite number of coefficients.

Next consider the waveforms used for channel transmission. The energy used over any finite interval T is limited both by regulatory agencies and by physical constraints on transmitters and antennas. One could consider waveforms of finite power but infinite duration and energy (such as the lowly sinusoid). On one hand, physical waveforms do not last forever (transmitters wear out or become obsolete), but on the other hand, *models* of physical waveforms can have infinite duration, modeling physical lifetimes that are much longer than any time scale of communication interest. Nonetheless, for reasons that will gradually unfold, the channel waveforms in this text will almost always be restricted to finite energy.

There is another important reason for concentrating on finite-energy waveforms. Not only are they the appropriate models for source and channel waveforms, but they also have remarkably simple and general properties. These properties rely on an additional constraint called *measurability* which is explained in the following section. These finite-energy measurable functions are called $\mathcal{L}_2$ functions. When time-constrained, they *always* have Fourier series, and without a time constraint, they *always* have Fourier transforms. Perhaps the most important property, however, is that $\mathcal{L}_2$ functions can be treated essentially as conventional vectors (see Chapter 5).

One might question whether a limitation to finite-energy functions is too constraining. For example, a sinusoid is often used to model the carrier in passband communication, and sinusoids have infinite energy because of their infinite duration. As seen later, however, when a finite-energy baseband waveform is modulated by that sinusoid up to passband, the resulting passband waveform has finite energy.

As another example, the unit impulse (the Dirac delta function $\delta(t)$) is a generalized function used to model waveforms of unit area that are nonzero only in a narrow region around $t = 0$, narrow relative to all other intervals of interest. The impulse response of a linear-time-invariant filter is, of course, the response to a unit impulse; this response approximates the response to a physical waveform that is sufficiently narrow and has unit area. The energy in that physical waveform, however, grows wildly as the waveform becomes more narrow. A rectangular pulse of width ε and height $1/\varepsilon$, for example, has unit area for all $\varepsilon > 0$ but has energy $1/\varepsilon$, which approaches ∞ as $\varepsilon \to 0$. One could view the energy in a unit impulse as being either undefined or infinite, but in no way could view it as being finite.

To summarize, there are many useful waveforms outside the finite-energy class. Although they are not physical waveforms, they are useful models of physical waveforms where energy is not important. Energy is such an important aspect of source and channel waveforms, however, that such waveforms can safely be limited to the finite-energy class.

4.3 $\mathcal{L}_2$ functions and Lebesgue integration over $[-T/2, T/2]$

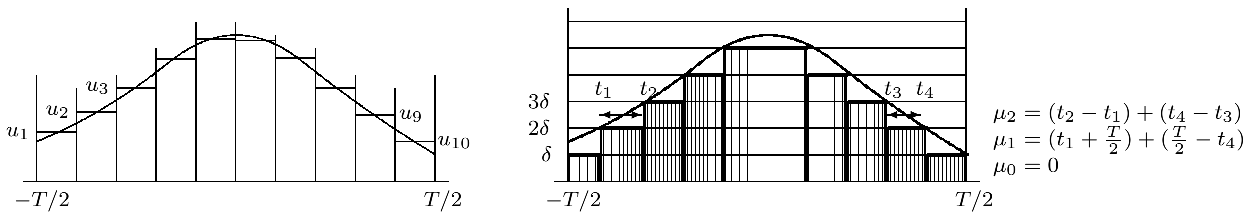
A function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is defined to be $\mathcal{L}_2$ if it is Lebesgue measurable and has a finite Lebesgue integral $\int_{-\infty}^{\infty} |u(t)|^2\, dt$. This section provides a basic and intuitive understanding of what these terms mean. The appendix provides proofs of the results, additional examples, and more depth of understanding. Still deeper understanding requires a good mathematics course in real and complex variables. The appendix is not required for basic engineering understanding of results in this and subsequent chapters, but it will provide deeper insight.

The basic idea of Lebesgue integration is no more complicated than the more common Riemann integration taught in freshman college courses. Whenever the Riemann integral exists, the Lebesgue integral also exists[10] and has the same value. Thus all the familiar ways of calculating integrals, including tables and numerical procedures, hold without change. The Lebesgue integral is more useful here, partly because it applies to a wider set of functions, but, more importantly, because it greatly simplifies the main results.

This section considers only time-limited functions, $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$. These are the functions of interest for Fourier series, and the restriction to a finite interval avoids some mathematical details better addressed later.

Figure 4.4 shows intuitively how Lebesgue and Riemann integration differ. Conventional Riemann integration of a nonnegative real-valued function $u(t)$ over an interval $[-T/2, T/2]$ is conceptually performed in Figure 4.4a by partitioning $[-T/2, T/2]$ into, say, i_0 intervals each of width T/i_0. The function is then approximated within the ith such interval by a single value u_i, such as the mid-point of values in the interval. The integral is then approximated as $\sum_{i=1}^{i_0}(T/i_0)u_i$. If the function is sufficiently smooth, then this approximation has a limit, called the Riemann integral, as $i_0 \to \infty$.

[10]There is a slight notional qualification to this which is discussed in the sinc function example of Section 4.5.1.



$\int_{-T/2}^{T/2} u(t)\,dt \approx \sum_{i=1}^{i_0} u_i/i_0$ $\qquad$ $\int_{-T/2}^{T/2} u(t)\,dt \approx \sum_m m\delta\,\mu_m$

(a): Riemann $\qquad$ (b): Lebesgue

Figure 4.4: Example of Riemann and Lebesgue integration

To integrate the same function by Lebesgue integration, the vertical axis is partitioned into intervals each of height δ, as shown in Figure 4.4(b). For the mth such interval,[11] $[m\delta, (m+1)\delta\,)$, let $\mathcal{E}_m$ be the set of values of t such that $m\delta \le u(t) < (m+1)\delta$. For example, the set $\mathcal{E}_2$ is illustrated by arrows in Figure 4.4 and is given by

$$\mathcal{E}_2 = \{t : 2\delta \le u(t) < 3\delta\} = [t_1, t_2) \cup (t_3, t_4].$$

As explained below, if $\mathcal{E}_m$ is a finite union of separated[12] intervals, its measure, μ_m is the sum of the widths of those intervals; thus μ_2 in the example above is given by

$$\mu_2 = \mu(\mathcal{E}_2) = (t_2 - t_1) + (t_4 - t_3). \tag{4.8}$$

Similarly, $\mathcal{E}_1 = [\frac{-T}{2}, t_1) \cup (t_4, \frac{T}{2}]$ and $\mu_1 = (t_1 + \frac{T}{2}) + (\frac{T}{2} - t_4)$.

The Lebesque integral is approximated as $\sum_m (m\delta)\mu_m$. This approximation is indicated by the vertically shaded area in the figure. The Lebesgue integral is essentially the limit as $\delta \to 0$.

In short, the Riemann approximation to the area under a curve splits the horizontal axis into uniform segments and sums the corresponding rectangular areas. The Lebesgue approximation splits the vertical axis into uniform segments and sums the height times width measure for each segment. In both cases, a limiting operation is required to find the integral, and Section 4.3.3 gives an example where the limit exists in the Lebesgue but not the Riemann case.

4.3.1 Lebesgue measure for a union of intervals

In order to explain Lebesgue integration further, measure must be defined for a more general class of sets.

The *measure* of an interval I from a to b, $a \le b$ is defined to be $\mu(I) = b - a \ge 0$. For any finite union of, say, ℓ separated intervals, $\mathcal{E} = \bigcup_{j=1}^{\ell} I_j$, the measure $\mu(\mathcal{E})$ is defined as

$$\mu(\mathcal{E}) = \sum_{j=1}^{\ell} \mu(I_j). \tag{4.9}$$

[11] The notation $[a,\, b)$ denotes the semiclosed interval $a \le t < b$. Similarly, $(a,\, b]$ denotes the semiclosed interval $a < t \le b$, $(a,\, b)$ the open interval $a < t < b$, and $[a,\, b]$ the closed interval $a \le t \le b$. In the special case where $a = b$, the interval $[a, a]$ consists of the single point a, whereas $[a, a), (a, a]$, and (a, a) are empty.

[12] Two intervals are *separated* if they are both nonempty and there is at least one point between them that lies in neither interval; *i.e.*, $(0, 1)$ and $(1, 2)$ are separated. In contrast, two sets are *disjoint* if they have no points in common. Thus $(0, 1)$ and $[1, 2]$ are disjoint but not separated.

This definition of $\mu(\mathcal{E})$ was used in (4.8) and is necessary for the approximation in Figure 4.4b to correspond to the area under the approximating curve. The fact that the measure of an interval does not depend on inclusion of the end points corresponds to the basic notion of area under a curve. Finally, since these separated intervals are all contained in $[-T/2, T/2]$, it is seen that the sum of their widths is at most T, *i.e.*,

$$0 \leq \mu(\mathcal{E}) \leq T. \tag{4.10}$$

Any finite union of, say, ℓ arbitrary intervals, $\mathcal{E} = \bigcup_{j=1}^{\ell} I_j$, can also be uniquely expressed as a finite union of at most ℓ separated intervals, say $I'_1, \dots, I'_k$, $k \leq \ell$ (see Exercise 4.5), and its measure is then given by

$$\mu(\mathcal{E}) = \sum_{j=1}^{k} \mu(I'_j). \tag{4.11}$$

The union of a countably infinite collection[13] of separated intervals, say $\mathcal{B} = \bigcup_{j=1}^{\infty} I_j$ is also defined to be measurable and has a measure given by

$$\mu(\mathcal{B}) = \lim_{\ell\to\infty} \sum_{j=1}^{\ell} \mu(I_j). \tag{4.12}$$

The summation on the right is bounded between 0 and T for each ℓ. Since $\mu(I_j) \geq 0$, the sum is nondecreasing in ℓ. Thus the limit exists and lies between 0 and T. Also the limit is independent of the ordering of the I_j (see Exercise 4.4).

Example 4.3.1. Let $I_j = (T2^{-2j}, T2^{-2j+1})$ for all integer $j \geq 1$. The jth interval then has measure $\mu(I_j) = 2^{-2j}$. These intervals get smaller and closer to 0 as j increases. They are easily seen to be separated. The union $\mathcal{B} = \bigcup_j I_j$ then has measure $\mu(\mathcal{B}) = \sum_{j=1}^{\infty} T2^{-2j} = T/3$. Visualize replacing the function in Figure 4.4 by one that oscillates faster and faster as $t \to 0$; $\mathcal{B}$ could then represent the set of points on the horizontal axis corresponding to a given vertical slice.

Example 4.3.2. As a variation of the above example, suppose $\mathcal{B} = \bigcup_j I_j$ where $I_j = [T2^{-2j}, T2^{-2j}]$ for each j. Then interval I_j consists of the single point $T2^{-2j}$ so $\mu(I_j) = 0$. In this case, $\sum_{j=1}^{\ell} \mu(I_j) = 0$ for each ℓ. The limit of this as $\ell \to \infty$ is also 0, so $\mu(\mathcal{B}) = 0$ in this case. By the same argument, the measure of any countably infinite set of points is 0.

Any countably infinite union of arbitrary (perhaps intersecting) intervals can be uniquely[14] represented as a *countable* (*i.e.*, either a countably infinite or finite) union of separated intervals (see Exercise 4.6); its measure is defined by applying (4.12) to that representation.

4.3.2 Measure for more general sets

It might appear that the class of countable unions of intervals is broad enough to represent any set of interest, but it turns out to be too narrow to allow the general kinds of statements that

[13] An elementary discussion of countability is given in Appendix 4A.1. Readers unfamiliar with ideas such as the countability of the rational numbers are strongly encouraged to read this appendix.

[14] The collection of separated intervals and the limit in (4.12) is unique, but the ordering of the intervals is not.

formed our motivation for discussing Lebesgue integration. One vital generalization is to require that the complement $\overline{\mathcal{B}}$ (relative to $[-T/2, T/2]$) of any measurable set $\mathcal{B}$ also be measurable.[15] Since $\mu([-T/2, T/2]) = T$ and every point of $[-T/2, T/2]$ lies in either $\mathcal{B}$ or $\overline{\mathcal{B}}$ but not both, the measure of $\overline{\mathcal{B}}$ should be $T - \mu(\mathcal{B})$. The reason why this property is necessary in order for the Lebesgue integral to correspond to the area under a curve is illustrated in Figure 4.5.

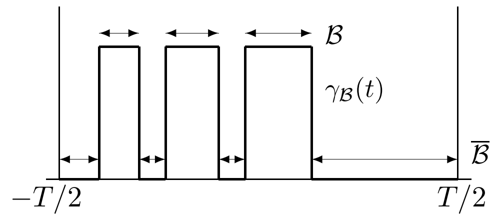


Figure 4.5: Let $f(t)$ have the value 1 on a set $\mathcal{B}$ and the value 0 elsewhere in $[-T/2, T/2]$. Then $\int f(t)\,dt = \mu(\mathcal{B})$. The complement $\overline{\mathcal{B}}$ of $\mathcal{B}$ is also illustrated and it is seen that $1 - f(t)$ is 1 on the set $\overline{\mathcal{B}}$ and 0 elsewhere. Thus $\int [1 - f(t)]\,dt = \mu(\overline{\mathcal{B}})$, which must equal $T - \mu(\mathcal{B})$ for integration to correspond to the area under a curve.

The *subset inequality* is another property that measure should have: this states that if $\mathcal{A}$ and $\mathcal{B}$ are both measurable and $\mathcal{A} \subseteq \mathcal{B}$, then $\mu(\mathcal{A}) \le \mu(B)$. One can also visualize from Figure 4.5 why this subset inequality is necessary for integration to represent the area under a curve.

Before defining which sets in $[-T/2, T/2]$ are measurable and which are not, a measure-like function called *outer measure* is introduced that exists for *all* sets in $[-T/2, T/2]$. For an arbitrary set $\mathcal{A}$, the set $\mathcal{B}$ is said to *cover* $\mathcal{A}$ if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ is a countable union of intervals. The outer measure $\mu^{\rm o}(\mathcal{A})$ is then essentially the measure of the smallest cover of $\mathcal{A}$. In particular,[16]

$$\mu^{\rm o}(\mathcal{A}) \;=\; \inf_{\mathcal{B}:\, \mathcal{B} \text{ covers } \mathcal{A}} \mu(\mathcal{B}). \tag{4.13}$$

Not surprisingly, the outer measure of a countable union of intervals is equal to its measure as already defined (see Appendix 4A.3).

Measurable sets and measure over the interval $[-T/2, T/2]$ can now be defined as follows:

Definition: A set $\mathcal{A}$ (over $[-T/2, T/2]$) is *measurable* if $\mu^{\rm o}(\mathcal{A}) + \mu^{\rm o}(\overline{\mathcal{A}}) = T$. If $\mathcal{A}$ is measurable, then its *measure*, $\mu(\mathcal{A})$, is the outer measure $\mu^{\rm o}(\mathcal{A})$.

Intuitively, then, a set is measurable if the set and its complement are sufficiently untangled that each can be covered by countable unions of intervals which have arbitrarily little overlap. The example at the end of Section 4A.4 constructs the simplest nonmeasurable set we are aware of; it should be noted how bizarre it is and how tangled it is with its complement.

The definition of measurability is a 'mathematician's definition' in the sense that it is very

[15] Appendix 4A.1 uses the set of rationals in $[-T/2, T/2]$ to illustrate that the complement $\overline{\mathcal{B}}$ of a countable union of intervals $\mathcal{B}$ need not be a countable union of intervals itself. In this case $\mu(\overline{\mathcal{B}}) = T - \mu(\mathcal{B})$, which is shown to be valid also when $\overline{\mathcal{B}}$ is a countable union of intervals.

[16] The infimum (inf) of a set of real numbers is essentially the minimum of that set. The difference between the minimum and the infimum can be seen in the example of the set of real numbers strictly greater than 1. This set has no minimum, since for each number in the set, there is a smaller number still greater than 1. To avoid this somewhat technical issue, the infimum is defined as the greatest lower bound of a set. In the example, all numbers less than or equal to 1 are lower bounds for the set, and 1 is then greatest lower bound, *i.e.*, the infimum. Every nonempty set of real numbers has an infimum if one includes $-\infty$ as a choice.

succinct and elegant, but doesn't provide many immediate clues about determining whether a set is measurable and, if so, what its measure is. This is now briefly discussd.

It is shown in Appendix 4A.3 that countable unions of intervals are measurable according to this definition, and the measure can be found by breaking the set into separated intervals. Also, by definition, the complement of every measurable set is also measurable, so the complements of countable unions of intervals are measurable. Next, if $\mathcal{A} \subseteq \mathcal{A}'$, then any cover of $\mathcal{A}'$ also covers $\mathcal{A}$ so the subset inequality is satisfied. This often makes it possible to find the measure of a set by using a limiting process on a sequence of measurable sets contained in or containing a set of interest. Finally, the following theorem is proven in Section 4A.4 of the appendix.

Theorem 4.3.1. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$, be any sequence of measurable sets. Then $\mathcal{S} = \bigcup_{j=1}^{\infty} \mathcal{A}_j$ and $\mathcal{D} = \bigcap_{j=1}^{\infty} \mathcal{A}_j$ are measurable. If $\mathcal{A}_1, \mathcal{A}_2, \ldots$ are also disjoint, then $\mu(\mathcal{S}) = \sum_j \mu(\mathcal{A}_j)$. If $\mu^{\mathrm{o}}(\mathcal{A}) = 0$, then $\mathcal{A}$ is measurable and has zero measure.*

This theorem and definition say that the collection of measurable sets is closed under countable unions, countable intersections, and complementation. This partly explains why it is so hard to find nonmeasurable sets and also why their existence can usually be ignored - they simply don't arise in the ordinary process of analysis.

Another consequence concerns sets of zero measure. It was shown earlier that any set containing only countably many points has zero measure, but there are many other sets of zero measure. The Cantor set example in Section 4A.4 illustrates a set of zero measure with uncountably many elements. The theorem implies that a set $\mathcal{A}$ has zero measure if, for any $\varepsilon > 0$, $\mathcal{A}$ has a cover $\mathcal{B}$ such that $\mu(\mathcal{B}) \leq \varepsilon$. The definition of measurability shows that the complement of any set of zero measure has measure T, *i.e.*, $[-T/2, T/2]$ is the cover of smallest measure. It will be seen shortly that for most purposes, including integration, sets of zero measure can be ignored and sets of measure T can be viewed as the entire interval $[-T/2, T/2]$.

This concludes our study of measurable sets on $[-T/2, T/2]$. The bottom line is that not all sets are measurable, but that non-measurable sets arise only from bizarre and artificial constructions and can usually be ignored. The definitions of measure and measurability might appear somewhat arbitrary, but in fact they arise simply through the natural requirement that intervals and countable unions of intervals be measurable with the given measure[17] and that the subset inequality and complement property be satisfied. If we wanted additional sets to be measurable, then at least one of the above properties would have to be sacrificed and integration itself would become bizarre. The major result here, beyond basic familiarity and intuition, is Theorem 4.3.1 which is used repeatedly in the following sections. The appendix fills in many important details and proves the results here

4.3.3 Measurable functions and integration over $[-T/2, T/2]$

A function $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$, is said to be *Lebesgue measurable* (or more briefly *measurable*) if the set of points $\{t : u(t) < \beta\}$ is measurable for each $\beta \in \mathbb{R}$. If $u(t)$ is measurable, then, as shown in Exercise 4.11, the sets $\{t : u(t) \leq \beta\}$, $\{t : u(t) \geq \beta\}$, $\{t : u(t) > \beta\}$ and $\{t : \alpha \leq u(t) < \beta\}$ are measurable for all $\alpha < \beta \in \mathbb{R}$. Thus, if a function is measurable, the

[17] We have not distinguished between the condition of being measurable and the actual measure assigned a set, which is natural for ordinary integration. The theory can be trivially generalized, however, to random variables restricted to $[-T/2, T/2]$. In this case, the measure of an interval is redefined to be the probability of that interval. Everything else remains the same except that some individual points might have non-zero probability.

measure $\mu_m = \mu(\{t : m\delta \le u(t) < (m{+}1)\delta\})$ associated with the mth horizontal slice in Figure 4.4 must exist for each $\delta > 0$ and m.

For the Lebesgue integral to exist, it is also necessary that the Figure 4.4 approximation to the Lebesgue integral has a limit as the vertical interval size δ goes to 0. Initially consider only nonnegative functions, $u(t) \ge 0$ for all t. For each integer $n \ge 1$, define the nth order approximation to the Lebesgue integal as that arising from partitioning the vertical axis into intervals each of height $\delta_n = 2^{-n}$. Thus a unit increase in n corresponds to halving the vertical interval size as illustrated below.

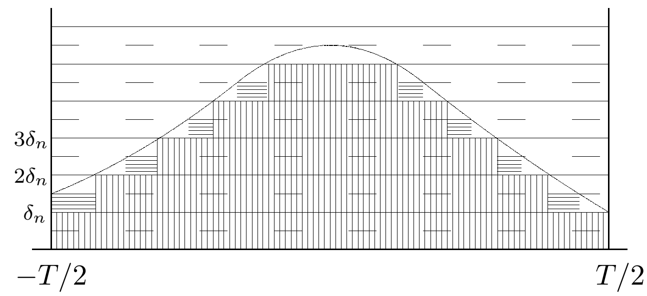


Figure 4.6: The improvement in the approximation to the Lebesgue integral by a unit increase in n is indicated by the horizontal crosshatching.

Let $\mu_{m,n}$ be the measure of $\{t : m2^{-n} \le u(t) < (m+1)2^{-n}\}$, *i.e.*, the measure of the set of $t \in [-T/2, T/2]$ for which $u(t)$ is in the mth vertical interval for the nth order approximation. The approximation $\sum_m m2^{-n}\,\mu_{m,n}$ might be infinite[18] for all n, and in this case the Lebesgue integral is said to be infinite. If the sum is finite for $n = 1$, however, the figure shows that the change in going from the approximation of order n to $n+1$ is nonnegative and upper bounded by $T2^{-n-1}$. Thus it is clear that the sequence of approximations has a finite limit which is defined[19] to be the *Lebesgue integral* of $u(t)$. In summary, the Lebesgue integral of an arbitrary measurable nonnegative function $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$ is finite if any approximation is finite and is then given by

$$\int u(t)\,dt = \lim_{n\to\infty} \sum_{m=0}^{\infty} m2^{-n}\mu_{m,n} \qquad \text{where} \quad \mu_{m,n} = \mu(t : m2^{-n} \le u(t) < (m+1)2^{-n}). \quad (4.14)$$

Example 4.3.3. Consider a function that has the value 1 for each rational number in $[-T/2, T/2]$ and 0 for all irrational numbers. The set of rationals has zero measure, as shown in Appendix 4A.1, so that each of the above approximations to the Lebesgue integral are 0 and thus the limit is zero. This is a simple example of a function that has a Lebesgue integral but no Riemann integral.

Next consider two non-negative measurable functions $u(t)$ and $v(t)$ on $[-T/2, T/2]$ and assume $u(t) = v(t)$ except on a set of zero measure. Then each of the approximations in (4.14) are identical for $u(t)$ and $v(t)$, and thus the two integrals are identical (either both infinite or both the same number). This same property will be seen to carry over for functions that also take on negative values and for complex valued functions. This property says that sets of zero measure

[18]For example, this sum is infinite if $u(t) = 1/|t|$ for $-T/2 \le t \le T/2$. The situation here is essentially the same for Riemann and Lebesgue integration.

[19]This limiting operation can be shown to be independent of how the quantization intervals approach 0.

can be ignored in integration. This is one of the major simplifications afforded by Lebesgue integration. Two functions that are the same except on a set of zero measure are said to be equal *almost everywhere*, abbreviated a.e. For example, the rectangular pulse and its Fourier series representation illustrated in Figure 4.2 are equal a.e.

For functions taking on both positive and negative values, the function $u(t)$ can be separated into a positive part $u^+(t)$ and a negative part $u^-(t)$. These are defined by

$$u^+(t) = \begin{cases} u(t) & \text{for } t : u(t) \geq 0 \\ 0 & \text{for } t : u(t) < 0 \end{cases}; \qquad u^-(t) = \begin{cases} 0 & \text{for } t : u(t) \geq 0 \\ -u(t) & \text{for } t : u(t) < 0. \end{cases}$$

For all $t \in [-T/2, T/2]$ then,

$$u(t) = u^+(t) - u^-(t). \tag{4.15}$$

If $u(t)$ is measurable, then $u^+(t)$ and $u^-(t)$ are also.[20] Since these are nonnegative, they can be integrated as before, and each integral exists with either a finite or infinite value. If at most one of these integrals is infinite, the Lebesgue integral of $u(t)$ is defined as

$$\int u(t) = \int u^+(t) - \int u^-(t)\, dt. \tag{4.16}$$

If both $\int u^+(t)\, dt$ and $\int u^-(t)\, dt$ are infinite, then the integral is undefined.

Finally, a complex function $\{u(t) : [-T/2\, T/2] \to \mathbb{C}\}$ is defined to be *measurable* if the real and imaginary parts of $u(t)$ are measurable. If the integrals of $\Re(u(t))$ and $\Im(u(t))$ are defined, then the Lebesgue integral $\int u(t)\, dt$ is defined by

$$\int u(t)\, dt = \int \Re(u(t))\, dt + i \int \Im(u(t))\, dt. \tag{4.17}$$

The integral is undefined otherwise. Note that this implies that any integration property of complex valued functions $\{u(t) : [-T/2\, T/2] \to \mathbb{C}\}$ is also shared by real valued functions $\{u(t) : [-T/2\, T/2] \to \mathbb{R}\}$.

4.3.4 Measurability of functions defined by other functions

The definitions of measurable functions and Lebesgue integration in the last subsection were quite simple given the concept of measure. However, functions are often defined in terms of other more elementary functions, so the question arises whether measurability of those elementary functions implies that of the defined function. The bottom-line answer is almost invariably yes. For this reason it is often assumed in the following sections that all functions of interest are measurable. Several results are now given fortifying this bottom-line view.

First, if $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$ is measurable, then $-u(t)$, $|u(t)|$, $u^2(t)$, $e^{u(t)}$, and $\ln |u(t)|$ are also measurable. These and similar results follow immediately from the definition of measurable functions and are derived in Exercise 4.12.

Next, if $u(t)$ and $v(t)$ are measurable, then $u(t) + v(t)$ and $u(t)v(t)$ are measurable (see Exercise 4.13).

[20] To see this, note that for $\beta > 0$, $\{t : u^+(t) < \beta\} = \{t : u(t) < \beta\}$. For $\beta \leq 0$, $\{t : u^+(t) < \beta\}$ is the empty set. A similar argument works for $u^-(t)$.

Finally, if $\{u_k(t) : [-T/2, T/2] \to \mathbb{R}\}$ is a measurable function for each integer $k \geq 1$, then $\inf_k u_k(t)$ is measurable. This can be seen by noting that $\{t : \inf_k[u_k(t)] \leq \alpha\} = \bigcup_k \{t : u_k(t) \leq \alpha\}$, which is measurable for each α. Using this result, Exercise 4.15, shows that $\lim_k u_k(t)$ is measurable if the limit exists for all $t \in [-T/2, T/2]$.

4.3.5 $\mathcal{L}_1$ and $\mathcal{L}_2$ functions over $[-T/2, T/2]$

A function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is said to be $\mathcal{L}_1$, or in the class $\mathcal{L}_1$, if $u(t)$ is measurable and the Lebesgue integral of $|u(t)|$ is finite.[21]

For the special case of a real function, $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$, the magnitude $|u(t)|$ can be expressed in terms of the positive and negative parts of $u(t)$ as $|u(t)| = u^+(t) + u^-(t)$. Thus $u(t)$ is $\mathcal{L}_1$ if and only if both $u^+(t)$ and $u^-(t)$ have finite integrals. In other words, $u(t)$ is $\mathcal{L}_1$ if and only if the Lebesgue integral of $u(t)$ is defined and finite.

For a complex function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$, it can be seen that $u(t)$ is $\mathcal{L}_1$ if and only if both $\Re[u(t)]$ and $\Im[u(t)]$ are $\mathcal{L}_1$. Thus $u(t)$ is $\mathcal{L}_1$ if and only if $\int u(t)\, dt$ is defined and finite.

A function $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$ or $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is said to be an $\mathcal{L}_2$ function, or a *finite-energy function*, if $u(t)$ is measurable and the Lebesgue integral of $|u(t)|^2$ is finite. All source and channel waveforms discussed in this text will be assumed to be $\mathcal{L}_2$. Although $\mathcal{L}_2$ functions are of primary interest here, the class of $\mathcal{L}_1$ functions is of almost equal importance in understanding Fourier series and Fourier transforms. An important relation between $\mathcal{L}_1$ and $\mathcal{L}_2$ is given in the following simple theorem, illustrated in Figure 4.7.

Theorem 4.3.2. *If $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is $\mathcal{L}_2$, then it is also $\mathcal{L}_1$.*

Proof: Note that $|u(t)| \leq |u(t)|^2$ for all t such that $|u(t)| \geq 1$. Thus $|u(t)| \leq |u(t)|^2 + 1$ for all t, so that $\int |u(t)|\, dt \leq \int |u(t)|^2\, dt + T$. If the function $u(t)$ is $\mathcal{L}_2$, then the right side of this equation is finite, so the function is also $\mathcal{L}_1$. □

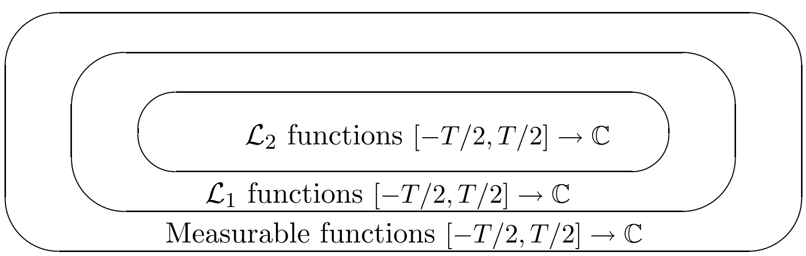


Figure 4.7: Illustration showing that for functions from $[-T/2, T/2]$ to $\mathbb{C}$, the class of $\mathcal{L}_2$ functions is contained in the class of $\mathcal{L}_1$ functions, which in turn is contained in the class of measurable functions. The restriction here to a finite domain such as $[-T/2, T/2]$ is necessary, as seen later.

This completes our basic introduction to measure and Lebesgue integration over the finite interval $[-T/2, T/2]$. The fact that the class of measurable sets is closed under complementation, countable unions, and countable intersections underlies the results about the measurability of

[21] $\mathcal{L}_1$ functions are sometimes called integrable functions.

functions being preserved over countable limits and sums. These in turn underlie the basic results about Fourier series, Fourier integrals, and orthogonal expansions. Some of those results will be stated without proof, but an understanding of measurability will let us understand what those results mean. Finally, ignoring sets of zero measure will simplify almost everything involving integration.

4.4 The Fourier series for $\mathcal{L}_2$ waveforms

The most important results about Fourier series for $\mathcal{L}_2$ functions are as follows:

Theorem 4.4.1 (Fourier series). *Let $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ be an $\mathcal{L}_2$ function. Then for each $k \in \mathbb{Z}$, the Lebesgue integral*

$$\hat{u}_k = \frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-2\pi ikt/T}\,dt \tag{4.18}$$

exists and satisfies $|\hat{u}_k| \le \frac{1}{T}\int |u(t)|\,dt < \infty$. Furthermore,

$$\lim_{\ell\to\infty}\int_{-T/2}^{T/2}\left|u(t) - \sum_{k=-\ell}^{\ell}\hat{u}_k e^{2\pi ikt/T}\right|^2 dt = 0, \tag{4.19}$$

where the limit is monotonic in ℓ. Also, the energy equation (4.6) is satisfied.

Conversely,, if $\{\hat{u}_k; k \in \mathbb{Z}\}$ is a two-sided sequence of complex numbers satisfying $\sum_{k=-\infty}^{\infty}|\hat{u}_k|^2 < \infty$, then an $\mathcal{L}_2$ function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ exists such that (4.6) and (4.19) are satisfied.

The first part of the theorem is simple. Since $u(t)$ is measurable and $e^{-2\pi ikt/T}$ is measurable for each k, the product $u(t)e^{-2\pi ikt/T}$ is measurable. Also $|u(t)e^{-2\pi ikt/T}| = |u(t)|$ so that $u(t)e^{-2\pi ikt/T}$ is $\mathcal{L}_1$ and the integral exists with the given upper bound (see Exercise 4.17). The rest of the proof is in the next chapter, Section 5.3.4.

The integral in (4.19) is the energy in the difference between $u(t)$ and the partial Fourier series using only the terms $-\ell \le k \le \ell$. Thus (4.19) asserts that $u(t)$ can be approximated arbitrarily closely (in terms of difference energy) by finitely many terms in its Fourier series.

A series is defined to *converge in $\mathcal{L}_2$* if (4.19) holds. The notation l.i.m. (limit in mean-square) is used to denote $\mathcal{L}_2$ convergence, so (4.19) is often abbreviated by

$$u(t) = \text{l.i.m.}\sum_k \hat{u}_k\, e^{2\pi ikt/T}\,\text{rect}(\frac{t}{T}). \tag{4.20}$$

The notation *does not* indicate that the sum in (4.20) converges pointwise to $u(t)$ at each t; for example, the Fourier series in Figure 4.2 converges to 1/2 rather than 1 at the values $t = \pm 1/4$. In fact, *any* two $\mathcal{L}_2$ functions that are equal a.e. have the same Fourier series coefficients. Thus the best to be hoped for is that $\sum_k \hat{u}_k\, e^{2\pi ikt/T}\,\text{rect}(\frac{t}{T})$ converges pointwise and yields a 'canonical representative' for all the $\mathcal{L}_2$ functions that have the given set of Fourier coefficients, $\{\hat{u}_k; k \in \mathbb{Z}\}$.

Unfortunately, there are some rather bizarre $\mathcal{L}_2$ functions (see the everywhere discontinuous example in Section 5A.1) for which $\sum_k \hat{u}_k\, e^{2\pi ikt/T}\,\text{rect}(\frac{t}{T})$ diverges for some values of t.

There is an important theorem due to Carleson [3], however, stating that if $u(t)$ is $\mathcal{L}_2$, then $\sum_k \hat{u}_k\, e^{2\pi ikt/T}\, \mathrm{rect}(\frac{t}{T})$ converges almost everywhere on $[-T/2, T/2]$. Thus for any $\mathcal{L}_2$ function $u(t)$, with Fourier coefficients $\{\hat{u}_k : k \in \mathbb{Z}\}$, there is a well-defined function,

$$\tilde{u}(t) = \begin{cases} \sum_{k=-\infty}^{\infty} \hat{u}_k\, e^{2\pi ikt/T}\, \mathrm{rect}(\frac{t}{T}) & \text{if the sum converges} \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

Since the sum above converges a.e., the Fourier coefficients of $\tilde{u}(t)$ given by (4.18) agree with those in (4.21). Thus $\tilde{u}(t)$ can serve as a canonical representative for all the $\mathcal{L}_2$ functions with the same Fourier coefficients $\{\hat{u}_k; k \in \mathbb{Z}\}$. From the difference-energy equation (4.7), it follows that the difference between any two $\mathcal{L}_2$ functions with the same Fourier coefficients has zero energy. Two $\mathcal{L}_2$ functions whose difference has zero energy are said to be $\mathcal{L}_2$ *equivalent*; thus all $\mathcal{L}_2$ functions with the same Fourier coefficients are $\mathcal{L}_2$ equivalent. Exercise 4.18 shows that two $\mathcal{L}_2$ functions are $\mathcal{L}_2$ equivalent if and only if they are equal almost everywhere.

In summary, each $\mathcal{L}_2$ function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ belongs to an equivalence class consisting of all $\mathcal{L}_2$ functions with the same set of Fourier coefficients. Each pair of functions in this equivalence class are $\mathcal{L}_2$ equivalent and equal a.e. The canonical represention in (4.21) is determined solely by the Fourier coefficients and is uniquely defined for any given set of Fourier coefficients satisfying $\sum_k |\hat{u}_k|^2 < \infty$; the corresponding equivalence class consists of the $\mathcal{L}_2$ functions that are equal to $\tilde{u}(t)$ a.e.

From an engineering standpoint, the sequence of ever closer approximations in (4.19) is usually more relevant than the notion of an equivalence class of functions with the same Fourier coefficients. In fact, for physical waveforms, there is no physical test that can distinguish waveforms that are $\mathcal{L}_2$ equivalent, since any such physical test requires an energy difference. At the same time, if functions $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ are consistently represented by their Fourier coefficients, then equivalence classes can usually be ignored.

For all but the most bizarre $\mathcal{L}_2$ functions, the Fourier series converges everywhere to some function that is $\mathcal{L}_2$ equivalent to the original function, and thus, as with the points $t = \pm 1/4$ in the example of Figure 4.2, it is usually unimportant how one views the function at those isolated points. Occasionally, however, particularly when discussing sampling and vector spaces, the concept of equivalence classes becomes relevant.

4.4.1 The T-spaced truncated sinusoid expansion

There is nothing special about the choice of 0 as the center point of a time-limited function. For a function $\{v(t) : [\Delta - T/2, \Delta + T/2] \to \mathbb{C}\}$ centered around some arbitrary time Δ, the *shifted Fourier series* over that interval is[22]

$$v(t) = \mathrm{l.i.m.} \sum_k \hat{v}_k\, e^{2\pi ikt/T}\, \mathrm{rect}\left(\frac{t-\Delta}{T}\right), \qquad \text{where} \tag{4.22}$$

$$\hat{v}_k = \frac{1}{T}\int_{\Delta-T/2}^{\Delta+T/2} v(t) e^{-2\pi ikt/T}\, dt, \qquad -\infty < k < \infty. \tag{4.23}$$

To see this, let $u(t) = v(t + \Delta)$. Then $u(0) = v(\Delta)$ and $u(t)$ is centered around 0 and has a Fourier series given by (4.20) and (4.18). Letting $\hat{v}_k = \hat{u}_k e^{-2\pi ik\Delta/T}$ yields (4.22) and (4.23).

[22]Note that the Fourier relationship between the function $v(t)$ and the sequence $\{v_k\}$ depends implicitly on the interval T and the shift Δ.

The results about measure and integration are not changed by this shift in the time axis.

Next, suppose that some given function $u(t)$ is either not time-limited or limited to some very large interval. An important method for source coding is first to break such a function into segments, say of duration T, and then to encode each segment[23] separately. A segment can be encoded by expanding it in a Fourier series and then encoding the Fourier series coefficients.

Most voice compression algorithms use such an approach, usually breaking the voice waveform into 20 msec segments. Voice compression algorithms typically use the detailed structure of voice rather than simply encoding the Fourier series coefficients, but the frequency structure of voice is certainly important in this process. Thus understanding the Fourier series approach is a good first step in understanding voice compression.

The implementation of voice compression (as well as most signal processing techniques) usually starts with sampling at a much higher rate than the segment duration above. This sampling is followed by high-rate quantization of the samples, which are then processed digitally. Conceptually, however, it is preferable to work directly with the waveform and with expansions such as the Fourier series. The analog parts of the resulting algorithms can then be implemented by the standard techniques of high-rate sampling and digital signal processing.

Suppose that an $\mathcal{L}_2$ waveform $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is segmented into segments $u_m(t)$ of duration T. Expressing $u(t)$ as the sum of these segments,[24]

$$u(t) = \text{l.i.m.} \sum_m u_m(t), \qquad \text{where } u_m(t) = u(t)\,\text{rect}\left(\frac{t}{T} - m\right). \tag{4.24}$$

Expanding each segment $u_m(t)$ by the shifted Fourier series of (4.22) and (4.23):

$$u_m(t) = \text{l.i.m.} \sum_k \hat{u}_{k,m}\, e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T} - m\right), \qquad \text{where} \tag{4.25}$$

$$\hat{u}_{k,m} = \frac{1}{T} \int_{mT-T/2}^{mT+T/2} u_m(t)\, e^{-2\pi i k t/T}\, dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} u(t)\, e^{-2\pi i k t/T} \text{rect}\left(\frac{t}{T} - m\right) dt. \tag{4.26}$$

Combining (4.24) and (4.25),

$$u(t) = \text{l.i.m.} \sum_m \sum_k \hat{u}_{k,m}\, e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T} - m\right).$$

This expands $u(t)$ as a weighted sum[25] of doubly indexed functions

$$u(t) = \text{l.i.m.} \sum_m \sum_k \hat{u}_{k,m} \theta_{k,m}(t) \qquad \text{where} \quad \theta_{k,m}(t) = e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T} - m\right). \tag{4.27}$$

[23]Any engineer, experienced or not, when asked to analyze a segment of a waveform, will automatically shift the time axis s that 0 is either the beginning or the center of the waveform. The added complication here simply arises from looking at multiple segments together so as to represent the entire waveform.

[24]This sum double-counts the points at the ends of the segments, but this makes no difference in terms of $\mathcal{L}_2$ convergence. Exercise 4.22 treats the convergence in (4.24) and (4.28) more carefully.

[25]Exercise 4.21 shows why (4.27) (and similar later expressions) are independent of the order of the limits.

The functions $\theta_{k,m}(t)$ are orthogonal, since, for $m \neq m'$, the functions $\theta_{k,m}(t)$ and $\theta_{k',m'}(t)$ do not overlap, and, for $m = m'$ and $k \neq k'$, $\theta_{k,m}(t)$ and $\theta_{k',m}(t)$ are orthogonal as before. These functions, $\{\theta_{k,m}(t); k, m \in \mathbb{Z}\}$, are called the *$T$-spaced truncated sinusoids* and the expansion in (4.27) is called the *T-spaced truncated sinusoid expansion.*

The coefficients $\hat{u}_{k,m}$ are indexed by $k, m \in \mathbb{Z}$ and thus form a countable set.[26] This permits the conversion of an arbitrary $\mathcal{L}_2$ waveform into a countably infinite sequence of complex numbers, in the sense that the numbers can be found from the waveform, and the waveform can be reconstructed from the sequence, at least up to $\mathcal{L}_2$ equivalence.

The l.i.m. notation in (4.27) denotes $\mathcal{L}_2$ convergence; *i.e.*,

$$\lim_{n,\ell\to\infty} \int_{-\infty}^{\infty} \left| u(t) - \sum_{m=-n}^{n} \sum_{k=-\ell}^{\ell} \hat{u}_{k,m}\theta_{k,m}(t) \right|^2 dt = 0. \tag{4.28}$$

This shows that any given $u(t)$ can be approximated arbitrarily closely by a finite set of coefficients. In particular, each segment can be approximated by a finite set of coefficients, and a finite set of segments approximates the entire waveform (although the required number of segments and coefficients per segment clearly depend on the particular waveform).

For data compression, a waveform $u(t)$ represented by the coefficients $\{\hat{u}_{k,m}; k, m \in \mathbb{Z}\}$ can be compressed by quantizing each $\hat{u}_{k,m}$ into a representative $\hat{v}_{k,m}$. The energy equation (4.6) and the difference-energy equation (4.7) generalize easily to the T-spaced truncated sinusoid expansion as

$$\int_{-\infty}^{\infty} |u(t)|^2\,dt = T \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\hat{u}_{k,m}|^2, \tag{4.29}$$

$$\int_{-\infty}^{\infty} |u(t) - v(t)|^2\,dt = T \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\hat{u}_{k,m} - \hat{v}_{k,m}|^2. \tag{4.30}$$

As in Section 4.2.1, a finite set of coefficients should be chosen for compression and the remaining coefficients should be set to 0. The problem of compression (given this expansion) is then to decide how many coefficients to compress, and how many bits to use for each selected coefficient. This of course requires a probabilistic model for the coefficients; this issue is discussed later.

There is a practical problem with the use of T-spaced truncated sinusoids as an expansion to be used in data compression. The boundaries of the segments usually act like step discontinuities (as in Figure 4.3) and this leads to slow convergence over the Fourier coefficients for each segment. These discontinuities could be removed prior to taking a Fourier series, but the current objective is simply to illustrate one general approach for converting arbitrary $\mathcal{L}_2$ waveforms to sequences of numbers. Before considering other expansions, it is important to look at Fourier transforms.

4.5 Fourier transforms and $\mathcal{L}_2$ waveforms

The T-spaced truncated sinusoid expansion corresponds closely to our physical notion of frequency. For example, musical notes correspond to particular frequencies (and their harmonics),

[26]Example 4A.2 in Section 4A.1 explains why the doubly indexed set above is countable.

but these notes persist for finite durations and then change to notes at other frequencies. However, the parameter T in the T-spaced expansion is arbitrary, and quantizing frequencies in increments of $1/T$ is awkward.

The Fourier transform avoids the need for segmentation into T-spaced intervals, but also removes the capability of looking at frequencies that change in time. It maps a function of time, $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ into a function of frequency,[27] $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$. The inverse Fourier transform maps $\hat{u}(f)$ back into $u(t)$, essentially making $\hat{u}(f)$ an alternative representation of $u(t)$.

The Fourier transform and its inverse are defined by

$$\hat{u}(f) = \int_{-\infty}^{\infty} u(t)e^{-2\pi ift}\,dt. \tag{4.31}$$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(f)e^{2\pi ift}\,df. \tag{4.32}$$

The time units are seconds and the frequency units Hertz (Hz), *i.e.*, cycles per second.

For now we take the conventional engineering viewpoint that any respectable function $u(t)$ has a Fourier transform $\hat{u}(f)$ given by (4.31), and that $u(t)$ can be retrieved from $\hat{u}(f)$ by (4.32). This will shortly be done more carefully for $\mathcal{L}_2$ waveforms.

The following table reviews a few standard Fourier transform relations. In the table, $u(t)$ and $\hat{u}(f)$ denote a Fourier transform pair, written $u(t) \leftrightarrow \hat{u}(f)$ and similarly $v(t) \leftrightarrow \hat{v}(f)$.

$$au(t) + bv(t) \leftrightarrow a\hat{u}(f) + b\hat{v}(f) \quad \text{linearity} \tag{4.33}$$

$$u^*(-t) \leftrightarrow \hat{u}^*(f) \quad \text{conjugation} \tag{4.34}$$

$$\hat{u}(t) \leftrightarrow u(-f) \quad \text{time/frequency duality} \tag{4.35}$$

$$u(t-\tau) \leftrightarrow e^{-2\pi if\tau}\hat{u}(f) \quad \text{time shift} \tag{4.36}$$

$$u(t)\,e^{2\pi if_0t} \leftrightarrow \hat{u}(f-f_0) \quad \text{frequency shift} \tag{4.37}$$

$$u(t/T) \leftrightarrow T\,\hat{u}(fT) \quad \text{scaling (for } T>0) \tag{4.38}$$

$$du(t)/dt \leftrightarrow 2\pi if\hat{u}(f) \quad \text{differentiation} \tag{4.39}$$

$$\int_{-\infty}^{\infty} u(\tau)v(t-\tau)\,d\tau \leftrightarrow \hat{u}(f)\hat{v}(f) \quad \text{convolution} \tag{4.40}$$

$$\int_{-\infty}^{\infty} u(\tau)v^*(\tau-t)\,d\tau \leftrightarrow \hat{u}(f)\hat{v}^*(f) \quad \text{correlation} \tag{4.41}$$

These relations will be used extensively in what follows. Time-frequency duality is particularly important, since it permits the translation of results about Fourier transforms to inverse Fourier transforms and *vice versa*.

Exercise 4.23 reviews the convolution relation (4.40). Equation (4.41) results from conjugating $\hat{v}(f)$ in (4.40).

Two useful special cases of any Fourier transform pair are:

$$u(0) = \int_{-\infty}^{\infty} \hat{u}(f)\,df; \tag{4.42}$$

$$\hat{u}(0) = \int_{-\infty}^{\infty} u(t)\,dt. \tag{4.43}$$

[27]The notation $\hat{u}(f)$, rather the more usual $U(f)$, is used here since capitalization is used to distinguish random variables from sample values. Later, $\{U(t) : \mathbb{R} \to \mathbb{C}\}$will be used to denote a random process, where, for each t, $U(t)$ is a random variable.

These are useful in checking multiplicative constants. Also *Parseval's theorem* results from applying (4.42) to (4.41):

$$\int_{-\infty}^{\infty} u(t)v^*(t)\,dt = \int_{-\infty}^{\infty} \hat{u}(f)\hat{v}^*(f)\,df. \tag{4.44}$$

As a corollary, replacing $v(t)$ by $u(t)$ in (4.44) results in the *energy equation* for Fourier transforms, namely

$$\int_{-\infty}^{\infty} |u(t)|^2\,dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2\,df. \tag{4.45}$$

The magnitude squared of the frequency function, $|\hat{u}(f)|^2$, is called the *spectral density* of $u(t)$. It is the energy per unit frequency (for positive and negative frequencies) in the waveform. The energy equation then says that energy can be calculated by integrating over either time or frequency.

As another corollary of (4.44), note that if $u(t)$ and $v(t)$ are orthogonal, then $\hat{u}(f)$ and $\hat{v}(f)$ are orthogonal; *i.e.*,

$$\int_{-\infty}^{\infty} u(t)v^*(t)\,dt = 0 \quad \text{if and only if} \quad \int_{-\infty}^{\infty} \hat{u}(f)\hat{v}^*(f)\,df = 0. \tag{4.46}$$

The following table gives a short set of useful and familiar transform pairs:

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \quad \leftrightarrow \quad \text{rect}(f) = \begin{cases} 1 & \text{for} \quad |f| \le 1/2 \\ 0 & \text{for} \quad |f| > 1/2 \end{cases} \tag{4.47}$$

$$e^{-\pi t^2} \quad \leftrightarrow \quad e^{-\pi f^2} \tag{4.48}$$

$$e^{-at};\ t \ge 0 \quad \leftrightarrow \quad \frac{1}{a + 2\pi i f} \qquad \text{for } a > 0 \tag{4.49}$$

$$e^{-a|t|} \quad \leftrightarrow \quad \frac{2a}{a^2 + (2\pi i f)^2} \qquad \text{for } a > 0 \tag{4.50}$$

The above table, in conjunction with the relations above, yields a large set of transform pairs. Much more extensive tables are widely available.

4.5.1 Measure and integration over $\mathbb{R}$

A set $\mathcal{A} \subseteq \mathbb{R}$ is defined to be *measurable* if $\mathcal{A} \cap [-T/2, T/2]$ is measurable for all $T > 0$. The definitions of measurability and measure in section 4.3.2 were given in terms of an overall interval $[-T/2, T/2]$, but Exercise 4.14 verifies that those definitions are in fact independent of T. That is, if $\mathcal{D} \subseteq [-T/2, T/2]$, is measurable relative to $[-T/2, T/2]$, then $\mathcal{D}$ is measurable relative to $[-T_1/2, T_1/2]$ for each $T_1 > T$ and $\mu(D)$ is the same relative to each of those intervals. Thus measure is defined unambiguously for all sets of bounded duration.

For an arbitrary measurable set $\mathcal{A} \in \mathbb{R}$, the measure of $\mathcal{A}$ is defined to be

$$\mu(\mathcal{A}) = \lim_{T\to\infty} \mu(\mathcal{A} \cap [-T/2, T/2]). \tag{4.51}$$

Since $\mathcal{A} \cap [-T/2, T/2]$ is increasing in T, the subset inequality says that $\mu(\mathcal{A} \cap [-T/2, T/2])$ is also increasing, so the limit in (4.51) must exist as either a finite or infinite value. For example, if $\mathcal{A}$ is taken to be $\mathbb{R}$ itself, then $\mu(\mathbb{R} \cap [-T/2, T/2]) = T$ and $\mu(\mathbb{R}) = \infty$. The possibility for measurable sets to have infinite measure is the primary difference between measure over $[-T/2, T/2]$ and $\mathbb{R}$.[28]

Theorem 4.3.1 carries over without change to sets defined over $\mathbb{R}$. Thus the collection of measurable sets over $\mathbb{R}$ is closed under countable unions and intersections. The measure of a measurable set might be infinite in this case, and if a set has finite measure, then its complement (over $\mathbb{R}$) must have infinite measure.

A real function $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ is *measurable* if the set $\{t : u(t) \le \beta\}$ is measurable for each $\beta \in \mathbb{R}$. Equivalently, $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ is measurable if and only if $u(t)\text{rect}(t/T)$ is measurable for all $T > 0$. A complex function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is measurable if the real and imaginary parts of $u(t)$ are measurable.

If $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ is measurable and nonnegative, there are two approaches to its Lebesgue integral. The first is to use (4.14) directly and the other is to first evaluate the integral over $[-T/2, T/2]$ and then go to the limit $T \to \infty$. Both approaches give the same result.[29]

For measurable real functions $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ that take on both positive and negative values, the same approach as in the finite duration case is successful. That is, let $u^+(t)$ and $u^-(t)$ be the positive and negative parts of $u(t)$ respectively. If at most one of these has an infinite integral, the integral of $u(t)$ is defined and has the value

$$\int u(t)\,dt = \int u^+(t)\,dt - \int u^-(t)\,dt.$$

Finally, a complex function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is defined to be *measurable* if the real and imaginary parts of $u(t)$ are measurable. If the integral of $\Re(u(t))$ and that of $\Im(u(t))$ are defined, then

$$\int u(t)\,dt = \int \Re(u(t))\,dt + i \int \Im(u(t))\,dt. \tag{4.52}$$

A function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is said to be in the class $\mathcal{L}_1$ if $u(t)$ is measurable and the Lebesgue integral of $|u(t)|$ is finite. As with integration over a finite interval, an $\mathcal{L}_1$ function has real and imaginary parts both of which are $\mathcal{L}_1$. Also the positive and negative parts of those real and imaginary parts have finite integrals.

Example 4.5.1. The sinc function, $\text{sinc}(t) = \sin(\pi t)/\pi t$ is sketched below and provides an interesting example of these definitions. Since $\text{sinc}(t)$ approaches 0 with increasing t only as $1/t$, the Riemann integral of $|\text{sinc}(t)|$ is infinite, and with a little thought it can be seen that the Lebesgue integral is also infinite. Thus $\text{sinc}(t)$ is not an $\mathcal{L}_1$ function. In a similar way, $\text{sinc}^+(t)$ and $\text{sinc}^-(t)$ have infinite integrals and thus the Lebesgue integral of $\text{sinc}(t)$ over $(-\infty, \infty)$ is undefined.

The Riemann integral in this case is said to be improper, but can still be calculated by integrating from $-A$ to $+A$ and then taking the limit $A \to \infty$. The result of this integration is 1, which is most easily found through the Fourier relationship (4.47) combined with (4.43). Thus, in a sense, the sinc function is an example where the Riemann integral exists but the Lebesgue integral does not. In a deeper sense, however, the issue is simply one of definitions; one can

[28] In fact, it was the restriction to finite measure that permitted the simple definition of measurability in terms of sets and their complements in Subsection 4.3.2.

[29] As explained shortly in the sinc function example, this is not necessarily true for functions taking on positive and negative values.

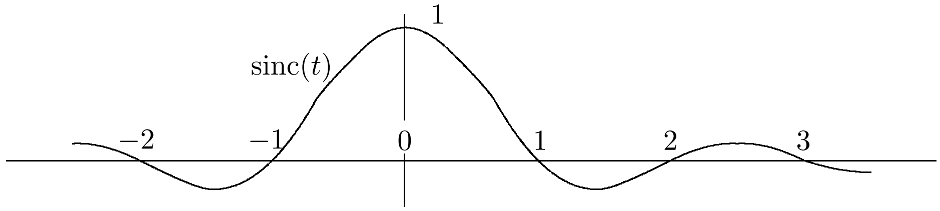


Figure 4.8: The function $\text{sinc}(t)$ goes to 0 as $1/t$ with increasing t

always use Lebesgue integration over $[-A, A]$ and go to the limit $A \to \infty$, getting the same answer as the Riemann integral provides.

A function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is said to be in the class $\mathcal{L}_2$ if $u(t)$ is measurable and the Lebesgue integral of $|u(t)|^2$ is finite. All source and channel waveforms will be assumed to be $\mathcal{L}_2$. As pointed out earlier, any $\mathcal{L}_2$ function of finite duration is also $\mathcal{L}_1$. $\mathcal{L}_2$ functions of infinite duration, however, need not be $\mathcal{L}_1$; the sinc function is a good example. Since $\text{sinc}(t)$ decays as $1/t$, it is not $\mathcal{L}_1$. However, $|\text{sinc}(t)|^2$ decays as $1/t^2$ as $t \to \infty$, so the integral is finite and $\text{sinc}(t)$ is an $\mathcal{L}_2$ function.

In summary, measure and integration over $\mathbb{R}$ can be treated in essentially the same way as over $[-T/2, T/2]$. The point sets and functions of interest can be truncated to $[-T/2, T/2]$ with a subsequent passage to the limit $T \to \infty$. As will be seen, however, this requires some care with functions that are not $\mathcal{L}_1$.

4.5.2 Fourier transforms of $\mathcal{L}_2$ functions

The Fourier transform does not exist for all functions, and when the Fourier transform does exist, there is not necessarily an inverse Fourier transform. This section first discusses $\mathcal{L}_1$ functions and then $\mathcal{L}_2$ functions. A major result is that $\mathcal{L}_1$ functions always have well-defined Fourier transforms, but the inverse transform does not always have very nice properties. $\mathcal{L}_2$ functions also always have Fourier transforms, but only in the sense of $\mathcal{L}_2$ equivalence. Here however, the inverse transform also exists in the sense of $\mathcal{L}_2$ equivalence. We are primarily interested in $\mathcal{L}_2$ functions, but the results about $\mathcal{L}_1$ functions will help in understanding the $\mathcal{L}_2$ transform.

Lemma 4.5.1. *Let $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ be $\mathcal{L}_1$. Then $\hat{u}(f) = \int_{-\infty}^{\infty} u(t)e^{-2\pi i f t}\, dt$ both exists and satisfies $|\hat{u}(f)| \leq \int |u(t)|\, dt$ for each $f \in \mathbb{R}$. Furthermore, $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ is a continuous function of f.*

Proof: Note that $|u(t)e^{-2\pi i f t}| = |u(t)|$ for all real t and f. Thus $u(t)e^{-2\pi i f t}$ is $\mathcal{L}_1$ for each f and the integral exists and satisfies the given bound. This is the same as the argument about Fourier series coefficients in Theorem 4.4.1. The continuity follows from a simple ϵ/δ argument (see Exercise 4.24). □

As an example, the function $u(t) = \text{rect}(t)$ is $\mathcal{L}_1$ and its Fourier transform, defined at each f, is the continuous function $\text{sinc}(f)$. As discussed before, $\text{sinc}(f)$ is not $\mathcal{L}_1$. The inverse transform of $\text{sinc}(f)$ exists at all t, equaling $\text{rect}(t)$ except at $t = \pm 1/2$, where it has the value $1/2$. Lemma 4.5.1 also applies to inverse transforms and verifies that $\text{sinc}(f)$ can not be $\mathcal{L}_1$, since its inverse transform is discontinuous.

Next consider $\mathcal{L}_2$ functions. It will be seen that the pointwise Fourier transform $\int u(t)e^{-2\pi ift}\,dt$ does not necessarily exist at each f, but that it does exist as an $\mathcal{L}_2$ limit. In exchange for this added complexity, however, the inverse transform exists in exactly the same sense. This result is called Plancherel's theorem and has a nice interpretation in terms of approximations over finite time and frequency intervals.

For any $\mathcal{L}_2$ function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ and any positive number A, define $\hat{u}_A(f)$ as the Fourier transform of the truncation of $u(t)$ to $[-A, A]$; *i.e.*,

$$\hat{u}_A(f) = \int_{-A}^{A} u(t)e^{-2\pi ift}\,dt. \tag{4.53}$$

The function $u(t)\text{rect}(\frac{t}{2A})$ has finite duration and is thus $\mathcal{L}_1$. It follows that $\hat{u}_A(f)$ is continuous and exists for all f by the above lemma. One would normally expect to take the limit in (4.53) as $A \to \infty$ to get the Fourier transform $\hat{u}(f)$, but this limit does not necessarily exist for each f. Plancherel's theorem, however, asserts that this limit exists in the $\mathcal{L}_2$ sense. This theorem is proved in Section 5A.1.

Theorem 4.5.1 (Plancherel, part 1). *For any $\mathcal{L}_2$ function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$, an $\mathcal{L}_2$ function $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ exists satisfying both*

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} |\hat{u}(f) - \hat{u}_A(f)|^2\,df = 0 \tag{4.54}$$

and the energy equation, (4.45).

This not only guarantees the existence of a Fourier transform (up to $\mathcal{L}_2$ equivalence), but also guarantees that it is arbitrarily closely approximated (in difference energy) by the continuous Fourier transforms of the truncated versions of $u(t)$. Intuitively what is happening here is that $\mathcal{L}_2$ functions must have an arbitrarily large fraction of their energy within sufficiently large truncated limits; the part of the function outside of these limits cannot significantly affect the $\mathcal{L}_2$ convergence of the Fourier transform.

The inverse transform is treated very similarly. For any $\mathcal{L}_2$ function $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ and any B, $0<B<\infty$, define

$$u_B(t) = \int_{-B}^{B} \hat{u}(f)e^{2\pi ift}\,df. \tag{4.55}$$

As before, $u_B(t)$ is a continuous $\mathcal{L}_2$ function for all B, $0<B<\infty$. The final part of Plancherel's theorem is then:

Theorem 4.5.2 (Plancherel, part 2). *For any $\mathcal{L}_2$ function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ let $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ be the Fourier transform of Theorem 4.5.1 and let $u_B(t)$ satisfy (4.55). Then*

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} |u(t) - u_B(t)|^2\,dt = 0. \tag{4.56}$$

The interpretation is similar to the first part of the theorem. Specifically the inverse transforms of finite frequency truncations of the transform are continuous and converge to an $\mathcal{L}_2$ limit as $B \to \infty$. It also says that this $\mathcal{L}_2$ limit is equivalent to the original function $u(t)$.

Using the limit in mean-square notation, both parts of the Plancherel theorem can be expressed by stating that every $\mathcal{L}_2$ function $u(t)$ has a Fourier transform $\hat{u}(f)$ satisfying

$$\hat{u}(f) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} u(t)e^{-2\pi ift}\,dt; \qquad u(t) = \underset{B\to\infty}{\text{l.i.m.}} \int_{-B}^{B} \hat{u}(f)e^{2\pi ift}\,df;$$

i.e., the inverse Fourier transform of $\hat{u}(f)$ is $\mathcal{L}_2$ equivalent to $u(t)$. The first integral above converges pointwise if $u(t)$ is also $\mathcal{L}_1$, and in this case converges pointwise to a continuous function $\hat{u}(f)$. If $u(t)$ is not $\mathcal{L}_1$, then the first integral need not converge pointwise. The second integral behaves in the analogous way.

It may help in understanding the Plancherel theorem to interpret it in terms of finding Fourier transforms using Riemann integration. Riemann integration over an infinite region is defined as a limit over finite regions. Thus the Riemann version of the Fourier transform is shorthand for

$$\hat{u}(f) = \lim_{A\to\infty} \int_{-A}^{A} u(t)e^{-2\pi ift}\,dt = \lim_{A\to\infty} \hat{u}_A(f). \tag{4.57}$$

Thus the Plancherel theorem can be viewed as replacing the Riemann integral with a Lebesgue integral and replacing the pointwise limit (if it exists) in (4.57) with $\mathcal{L}_2$ convergence. The Fourier transform over the finite limits $-A$ to A is continuous and well-behaved, so the major difference comes in using $\mathcal{L}_2$ convergence as $A \to \infty$.

As an example of the Plancherel theorem, let $u(t) = \text{rect}(t)$. Then $\hat{u}_A(f) = \text{sinc}(f)$ for all $A \geq 1/2$, so $\hat{u}(f) = \text{sinc}(f)$. For the inverse transform, $u_B(t) = \int_{-B}^{B} \text{sinc}(f)\,df$ is messy to compute but can be seen to approach $\text{rect}(t)$ as $B \to \infty$ except at $t = \pm 1/2$, where it equals $1/2$. At $t = \pm 1/2$, the inverse transform is $1/2$, whereas $u(t) = 1$.

As another example, consider the function $u(t)$ where $u(t) = 1$ for rational values of $t \in [0, 1]$ and $u(t) = 0$ otherwise. Since this is 0 a.e, the Fourier transform $\hat{u}(f)$ is 0 for all f and the inverse transform is 0, which is $\mathcal{L}_2$ equivalent to $u(t)$. Finally, Example 5A.1 in Section 5A.1 illustrates a bizarre $\mathcal{L}_1$ function $g(t)$ that is everywhere discontinuous. Its transform $\hat{g}(f)$ is bounded and continuous by Lemma 4.5.1, but is not $\mathcal{L}_1$. The inverse transform is again discontinuous everywhere in $(0, 1)$ and unbounded over every subinterval. This example makes clear why the inverse transform of a continuous function of frequency might be bizarre, thus reinforcing our focus on $\mathcal{L}_2$ functions rather than a more conventional focus on notions such as continuity.

In what follows, $\mathcal{L}_2$ convergence, as in the Plancherel theorem, will be seen as increasingly friendly and natural. Regarding two functions whose difference has 0 energy as being the same (formally, as $\mathcal{L}_2$ equivalent) allows us to avoid many trivialities, such as how to define a discontinuous function at its discontinuities. In this case, engineering common-sense and sophisticated mathematics arrive at the same conclusion.

Finally, it can be shown that all the Fourier transform relations in (4.33) to (4.41) except differentiation hold for all $\mathcal{L}_2$ functions (see Exercises 4.26 and 5.15). The derivative of an $\mathcal{L}_2$ function need not be $\mathcal{L}_2$, and need not have a well-defined Fourier transform.

4.6 The DTFT and the sampling theorem

The discrete-time Fourier transform (DTFT) is the time/frequency dual of the Fourier series. It will be shown that the DTFT leads immediately to the sampling theorem.

4.6.1 The discrete-time Fourier transform

Let $\hat{u}(f)$ be an $\mathcal{L}_2$ function of frequency, nonzero only for $-\mathsf{W} \le f \le \mathsf{W}$. The DTFT of $\hat{u}(f)$ over $[-\mathsf{W}, \mathsf{W}]$ is then defined by

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k e^{-2\pi i k f/(2\mathsf{W})} \text{rect}\left(\frac{f}{2\mathsf{W}}\right), \tag{4.58}$$

where the DTFT coefficients $\{u_k; k \in \mathbb{Z}\}$ are given by

$$u_k = \frac{1}{2\mathsf{W}} \int_{-\mathsf{W}}^{\mathsf{W}} \hat{u}(f) e^{2\pi i k f/(2\mathsf{W})}\, df. \tag{4.59}$$

These are the same as the Fourier series equations, replacing t by f, T by $2\mathsf{W}$, and $e^{2\pi i \cdots}$ by $e^{-2\pi i \cdots}$. Note that $\hat{u}(f)$ has an inverse Fourier transform $u(t)$ which is thus baseband-limited to $[-\mathsf{W}, \mathsf{W}]$. As will be shown shortly, the sampling theorem relates the samples of this baseband waveform to the coefficients in (4.59).

The Fourier series theorem (Theorem 4.4.1) clearly applies to (4.58)-(4.59) with the above notational changes; it is repeated here for convenience.

Theorem 4.6.1 (DTFT). *Let $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ be an $\mathcal{L}_2$ function. Then for each $k \in \mathbb{Z}$, the Lebesgue integral (4.59) exists and satisfies $|u_k| \le \frac{1}{2W} \int |\hat{u}(f)|\, df < \infty$. Furthermore,*

$$\lim_{\ell \to \infty} \int_{-W}^{W} \left| \hat{u}(f) - \sum_{k=-\ell}^{\ell} u_k\, e^{-2\pi i k f/(2W)} \right|^2 df = 0, \qquad \textit{and} \tag{4.60}$$

$$\int_{-W}^{W} |\hat{u}(f)|^2\, df = 2W \sum_{k=-\infty}^{\infty} |u_k|^2. \tag{4.61}$$

Finally, if $\{u_k,\ k \in \mathbb{Z}\}$ is a sequence of complex numbers satisfying $\sum_k |u_k|^2 < \infty$, then an $\mathcal{L}_2$ function $\{\hat{u}(f) : [-W,\ W] \to \mathbb{C}\}$ exists satisfying (4.60) and (4.61).

As before, (4.58) is shorthand for (4.60). Again, this says that any desired approximation accuracy, in terms of energy, can be achieved by using enough terms in the series.

Both the Fourier series and the DTFT provide a one-to-one transformation (in the sense of $\mathcal{L}_2$ convergence) between a function and a sequence of complex numbers. In the case of the Fourier series, one usually starts with a function $u(t)$ and uses the sequence of coefficients to represent the function (up to $\mathcal{L}_2$ equivalence). In the case of the DTFT, one often starts with the sequence and uses the frequency function to represent the sequence. Since the transformation goes both ways, however, one can view the function and the sequence as equally fundamental.

4.6.2 The sampling theorem

The DTFT is next used to establish the sampling theorem, which in turn will help interpret the DTFT. The DTFT (4.58) expresses $\hat{u}(f)$ as a weighted sum of truncated sinusoids in frequency,

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k \hat{\phi}_k(f), \quad \text{where} \quad \hat{\phi}_k(f) = e^{-2\pi i k f/(2\mathsf{W})} \text{rect}\left(\frac{f}{2\mathsf{W}}\right). \tag{4.62}$$

Ignoring any questions of convergence for the time being, the inverse Fourier transform of $\hat{u}(f)$ is then given by $u(t) = \sum_k u_k \phi_k(t)$, where $\phi_k(t)$ is the inverse transform of $\hat{\phi}_k(f)$. Since the inverse transform[30] of $\text{rect}(\frac{f}{2\mathsf{W}})$ is $2\mathsf{W}\text{sinc}(2\mathsf{W}t)$, the time-shift relation implies that the inverse transform of $\hat{\phi}_k(f)$ is

$$\phi_k(t) = 2\mathsf{W}\text{sinc}(2\mathsf{W}t - k) \quad \leftrightarrow \quad \hat{\phi}_k(f) = e^{-2\pi i k f/(2\mathsf{W})}\text{rect}(\frac{f}{2\mathsf{W}}). \tag{4.63}$$

Thus $u(t)$, the inverse transform of $\hat{u}(f)$, is given by

$$u(t) = \sum_{k=-\infty}^{\infty} u_k \phi_k(t) = \sum_{k=-\infty}^{\infty} 2\mathsf{W} u_k \,\text{sinc}(2\mathsf{W}t - k). \tag{4.64}$$

Since the set of truncated sinusoids $\{\hat{\phi}_k; k \in \mathbb{Z}\}$ are orthogonal, the sinc functions $\{\phi_k; k \in \mathbb{Z}\}$ are also orthogonal from (4.46).

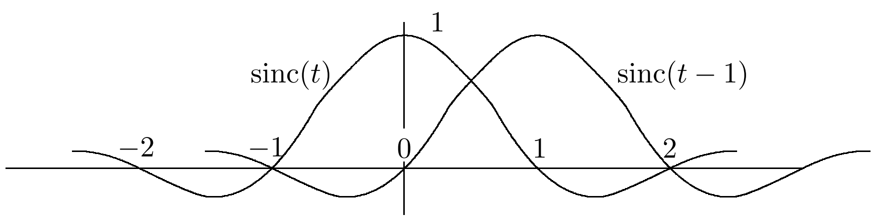


Figure 4.9: Sketch of $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ and $\text{sinc}(t-1)$. Note that these spaced sinc functions are orthogonal to each other.

Note that $\text{sinc}(t)$ equals 1 for $t = 0$ and 0 for all other integer t. Thus if (4.64) is evaluated for $t = \frac{k}{2\mathsf{W}}$, the result is that $u(\frac{k}{2\mathsf{W}}) = 2\mathsf{W}u_k$ for all integer k. Substituting this into (4.64) results in the equation known as the sampling equation,

$$u(t) = \sum_{k=-\infty}^{\infty} u(\frac{k}{2\mathsf{W}}) \,\text{sinc}(2\mathsf{W}t - k).$$

This says that a baseband-limited function is specified by its samples at intervals $T = 1/(2\mathsf{W})$. In terms of this sample interval, the sampling equation is

$$u(t) = \sum_{k=-\infty}^{\infty} u\,(kT)\,\text{sinc}(\frac{t}{T} - k). \tag{4.65}$$

The following theorem makes this precise. See Section 5A.2 for an insightful proof.

Theorem 4.6.2 (Sampling theorem). *Let $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ be a continuous $\mathcal{L}_2$ function baseband-limited to W. Then (4.65) specifies $u(t)$ in terms of its T-spaced samples with $T = \frac{1}{2\mathsf{W}}$. The sum in (4.65) converges to $u(t)$ for each $t \in \mathbb{R}$ and $u(t)$ is bounded at each t by $|u(t)| \le \int_{-\mathsf{W}}^{\mathsf{W}} |\hat{u}(f)|\,df < \infty$.*

The following example illustrates why $u(t)$ is assumed to be continuous above.

[30] This is the time/frequency dual of (4.47). $\hat{u}(f) = \text{rect}(\frac{f}{2\mathsf{W}})$ is both $\mathcal{L}_1$ and $\mathcal{L}_2$; $u(t)$ is continuous and $\mathcal{L}_2$ but not $\mathcal{L}_1$. From the Plancherel theorem, the transform of $u(t)$, in the $\mathcal{L}_2$ sense, is $\hat{u}(f)$.

Example 4.6.1 (A discontinuous baseband function). Let $u(t)$ be a continuous $\mathcal{L}_2$ baseband function limited to $|f| \leq 1/2$. Let $v(t)$ satisfy $v(t) = u(t)$ for all noninteger t and $v(t) = u(t) + 1$ for all integer t. Then $u(t)$ and $v(t)$ are $\mathcal{L}_2$ equivalent, but their samples at each integer time differ by 1. Their Fourier transforms are the same, say $\hat{u}(f)$, since the differences at isolated points have no effect on the transform. Since $\hat{u}(f)$ is nonzero only in $[-\mathsf{W}, \mathsf{W}]$, it is $\mathcal{L}_1$. According to the time/frequency dual of Lemma 4.5.1, the point-wise inverse Fourier transform of $\hat{u}(f)$ is a continuous function, say $u(t)$. Out of all the $\mathcal{L}_2$ equivalent waveforms that have the transform $\hat{u}(f)$, only $u(t)$ is continuous, and it is that $u(t)$ that satisfies the sampling theorem.

The function $v(t)$ is equal to $u(t)$ except for the isolated discontinuities at each integer point. One could view $v(t)$ as baseband-limited also, but $v(t)$ is clearly not physically meaningful and is not the continuous function of the theorem.

The above example illustrates an ambiguity about the meaning of baseband-limited functions. One reasonable definition is that an $\mathcal{L}_2$ function $u(t)$ is baseband-limited to W if $\hat{u}(f)$ is 0 for $|f| > \mathsf{W}$. Another reasonable definition is that $u(t)$ is baseband-limited to W if $u(t)$ is the pointwise inverse Fourier transform of a function $\hat{u}(f)$ that is 0 for $|f| > \mathsf{W}$. For a given $\hat{u}(f)$, there is a unique $u(t)$ according to the second definition and it is continuous; all the functions that are $\mathcal{L}_2$ equivalent to $u(t)$ are bandlimited by the first definition, and all but $u(t)$ are discontinuous and potentially violate the sampling equation. Clearly the second definition is preferable on both engineering and mathematical grounds.

Definition: An $\mathcal{L}_2$ function is *baseband-limited* to W if it is the pointwise inverse transform of an $\mathcal{L}_2$ function $\hat{u}(f)$ that is 0 for $|f| > \mathsf{W}$. Equivalently, it is baseband-limited to W if it is continuous and its Fourier transform is 0 for $|f| > 0$.

The DTFT can now be further interpreted. Any baseband-limited $\mathcal{L}_2$ function $\{\hat{u}(f) : [-\mathsf{W}, \mathsf{W}] \to \mathbb{C}\}$ has both an inverse Fourier transform $u(t) = \int \hat{u}(f) e^{2\pi i f t}\, df$ and a DTFT sequence given by (4.58). The coefficients u_k of the DTFT are the scaled samples, $Tu(kT)$, of $u(t)$, where $T = \frac{1}{2\mathsf{W}}$. Put in a slightly different way, the DTFT in (4.58) is the Fourier transform of the sampling equation (4.65) with $u(kT) = u_k/T$.[31]

It is somewhat surprising that the sampling theorem holds with pointwise convergence, whereas its transform, the DTFT, holds only in the $\mathcal{L}_2$ equivalence sense. The reason is that the function $\hat{u}(f)$ in the DTFT is $\mathcal{L}_1$ but not necessarily continuous, whereas its inverse transform $u(t)$ is necessarily continuous but not necessarily $\mathcal{L}_1$.

The set of functions $\{\hat{\phi}_k(f); k \in \mathbb{Z}\}$ in (4.63) is an orthogonal set, since the interval $[-\mathsf{W}, \mathsf{W}]$ contains an integer number of cycles from each sinusoid. Thus, from (4.46), the set of sinc functions in the sampling equation is also orthogonal. Thus both the DTFT and the sampling theorem expansion are orthogonal expansions. It follows (as will be shown carefully later) that the energy equation,

$$\int_{-\infty}^{\infty} |u(t)|^2\, dt = T \sum_{k=-\infty}^{\infty} |u(kT)|^2\,, \tag{4.66}$$

holds for any continuous $\mathcal{L}_2$ function $u(t)$ baseband-limited to $[-\mathsf{W}, \mathsf{W}]$ with $T = \frac{1}{2\mathsf{W}}$.

[31] Note that the DTFT is the time/frequency *dual* of the Fourier series but is the *Fourier transform* of the sampling equation.

In terms of source coding, the sampling theorem says that any $\mathcal{L}_2$ function $u(t)$ that is baseband-limited to W can be sampled at rate 2W (*i.e.*, at intervals $T = \frac{1}{2W}$) and the samples can later be used to perfectly reconstruct the function. This is slightly different from the channel coding situation where a sequence of signal values are mapped into a function from which the signals can later be reconstructed. The sampling theorem shows that any $\mathcal{L}_2$ baseband-limited function can be represented by its samples. The following theorem, proved in Section 5A.2, covers the channel coding variation:

Theorem 4.6.3 (Sampling theorem for transmission). *Let $\{a_k;\, k{\in}\mathbb{Z}\}$ be an arbitrary sequence of complex numbers satisfying $\sum_k |a_k|^2 < \infty$. Then $\sum_k a_k \operatorname{sinc}(2Wt - k)$ converges pointwise to a continuous bounded $\mathcal{L}_2$ function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ that is baseband-limited to W and satisfies $a_k = u(\frac{k}{2W})$ for each k.*

4.6.3 Source coding using sampled waveforms

The introduction and Figure 4.1 discuss the sampling of an analog waveform $u(t)$ and quantizing the samples as the first two steps in analog source coding. Section 4.2 discusses an alternative in which successive segments $\{u_m(t)\}$ of the source are each expanded in a Fourier series, and then the Fourier series coefficients are quantized. In this latter case, the received segments $\{v_m(t)\}$ are reconstructed from the quantized coefficients. The energy in $u_m(t) - v_m(t)$ is given in (4.7) as a scaled version of the sum of the squared coefficient differences. This section treats the analogous relationship when quantizing the samples of a baseband-limited waveform.

For a continuous function $u(t)$, baseband-limited to W, the samples $\{u(kT); k \in \mathbb{Z}\}$ at intervals $T = 1/(2W)$ specify the function. If $u(kT)$ is quantized to $v(kT)$ for each k, and $u(t)$ is reconstructed as $v(t) = \sum_k v(kT) \operatorname{sinc}(\frac{t}{T} - k)$, then, from (4.66), the mean-squared error is given by

$$\int_{-\infty}^{\infty} |u(t) - v(t)|^2 \, dt = T \sum_{k=-\infty}^{\infty} |u(kT) - v(kT)|^2. \tag{4.67}$$

Thus whatever quantization scheme is used to minimize the mean-squared error between a sequence of samples, that same strategy serves to minimize the mean-squared error between the corresponding waveforms.

The results in Chapter 3 regarding mean-squared distortion for uniform vector quantizers give the distortion at any given bit rate per sample as a linear function of the mean-squared value of the source samples. If any sample has an infinite mean-squared value, then either the quantization rate is infinite or the mean-squared distortion is infinite. This same result then carries over to waveforms. This starts to show why the restriction to $\mathcal{L}_2$ source waveforms is important. It also starts to show why general results about $\mathcal{L}_2$ waveforms are important.

The sampling theorem tells the story for sampling baseband-limited waveforms. However, physical source waveforms are not perfectly limited to some frequency W; rather, their spectra usually drop off rapidly above some nominal frequency W. For example, audio spectra start dropping off well before the nominal cutoff frequency of 4 kHz, but often have small amounts of energy up to 20 kHz. Then the samples at rate 2W do not quite specify the waveform, which leads to an additional source of error, called aliasing. Aliasing will be discussed more fully in the next two subsections.

There is another unfortunate issue with the sampling theorem. The sinc function is nonzero over all noninteger times. Recreating the waveform at the receiver[32] from a set of samples thus requires infinite delay. Practically, of course, sinc functions can be truncated, but the sinc waveform decays to zero as $1/t$, which is impractically slow. Thus the clean result of the sampling theorem is not quite as practical as it first appears.

4.6.4 The sampling theorem for $[\Delta - \mathsf{W}, \Delta + \mathsf{W}]$

Just as the Fourier series generalizes to time intervals centered at some arbitrary time Δ, the DTFT generalizes to frequency intervals centered at some arbitrary frequency Δ.

Consider an $\mathcal{L}_2$ frequency function $\{\hat{v}(f) : [\Delta{-}\mathsf{W},\ \Delta{+}\mathsf{W}] \to \mathbb{C}\}$. The *shifted DTFT* for $\hat{v}(f)$ is then

$$\hat{v}(f) = \text{l.i.m.} \sum_k v_k e^{-2\pi i k f/(2\mathsf{W})} \operatorname{rect}\left(\frac{f-\Delta}{2\mathsf{W}}\right) \qquad \text{where} \tag{4.68}$$

$$v_k = \frac{1}{2\mathsf{W}} \int_{\Delta-\mathsf{W}}^{\Delta+\mathsf{W}} \hat{v}(f) e^{2\pi i k f/(2\mathsf{W})}\, df. \tag{4.69}$$

Equation (4.68) is an orthogonal expansion,

$$\hat{v}(f) = \text{l.i.m.} \sum_k v_k \hat{\theta}_k(f) \quad \text{where} \quad \hat{\theta}_k(f) = e^{-2\pi i k f/(2\mathsf{W})} \operatorname{rect}\left(\frac{f-\Delta}{2\mathsf{W}}\right).$$

The inverse Fourier transform of $\hat{\theta}_k(f)$ can be calculated by shifting and scaling to be

$$\theta_k(t) = 2\mathsf{W} \operatorname{sinc}(2\mathsf{W}t - k)\, e^{2\pi i \Delta (t - \frac{k}{2\mathsf{W}})} \quad \leftrightarrow \quad \hat{\theta}_k(f) = e^{-2\pi i k f/(2\mathsf{W})} \operatorname{rect}\left(\frac{f-\Delta}{2\mathsf{W}}\right). \tag{4.70}$$

Let $v(t)$ be the inverse Fourier transform of $\hat{v}(f)$.

$$v(t) = \sum_k v_k \theta_k(t) = \sum_k 2\mathsf{W} v_k \operatorname{sinc}(2\mathsf{W}t - k)\, e^{2\pi i \Delta (t - \frac{k}{2\mathsf{W}})}.$$

For $t = \frac{k}{2\mathsf{W}}$, only the kth term above is nonzero, and $v(\frac{k}{2\mathsf{W}}) = 2\mathsf{W} v_k$. This generalizes the sampling equation to the frequency band $[\Delta{-}\mathsf{W},\ \Delta{+}\mathsf{W}]$,

$$v(t) = \sum_k v(\frac{k}{2\mathsf{W}}) \operatorname{sinc}(2\mathsf{W}t - k)\, e^{2\pi i \Delta (t - \frac{k}{2\mathsf{W}})}.$$

Defining the sampling interval $T = 1/(2\mathsf{W})$ as before, this becomes

$$v(t) = \sum_k v(kT) \operatorname{sinc}(\frac{t}{T} - k)\, e^{2\pi i \Delta (t - kT)}. \tag{4.71}$$

Theorems 4.6.2 and 4.6.3 apply to this more general case. That is, with $v(t) = \int_{\Delta-\mathsf{W}}^{\Delta+\mathsf{W}} \hat{v}(f) e^{2\pi i f t}\, df$, the function $v(t)$ is bounded and continuous and the series in (4.71) converges for all t. Similarly, if $\sum_k |v(kT)|^2 < \infty$, there is a unique continuous $\mathcal{L}_2$ function $\{v(t) : [\Delta{-}\mathsf{W},\ \Delta{+}\mathsf{W}] \to \mathbb{C}\}$, $\mathsf{W} = 1/(2T)$ with those sample values.

[32] Recall that the receiver time reference is delayed from that at the source by some constant τ. Thus $v(t)$, the receiver estimate of the source waveform $u(t)$ at source time t, is recreated at source time $t + \tau$. With the sampling equation, even if the sinc function is approximated, τ is impractically large.

4.7 Aliasing and the sinc-weighted sinusoid expansion

In this section an orthogonal expansion for arbitrary $\mathcal{L}_2$ functions called the *T-spaced sinc-weighted sinusoid expansion* is developed. This expansion is very similar to the T-spaced truncated sinusoid expansion discussed earlier, except that its set of orthogonal waveforms consist of time and frequency shifts of a sinc function rather than a rectangular function. This expansion is then used to discuss the important concept of degrees of freedom. Finally this same expansion is used to develop the concept of aliasing. This will help in understanding sampling for functions that are only approximately frequency-limited.

4.7.1 The T-spaced sinc-weighted sinusoid expansion

Let $u(t) \leftrightarrow \hat{u}(f)$ be an arbitrary $\mathcal{L}_2$ transform pair, and segment $\hat{u}(f)$ into intervals[33] of width $2\mathsf{W}$. Thus

$$\hat{u}(f) = \text{l.i.m.} \sum_m \hat{v}_m(f), \qquad \text{where} \quad \hat{v}_m(f) = \hat{u}(f)\,\text{rect}(\frac{f}{2\mathsf{W}} - m).$$

Note that $\hat{v}_0(f)$ is non-zero only in $[-\mathsf{W}, \mathsf{W}]$ and thus corresponds to an $\mathcal{L}_2$ function $v_0(t)$ baseband-limited to W. More generally, for arbitrary integer m, $\hat{v}_m(f)$ is non-zero only in $[\Delta{-}\mathsf{W},\ \Delta{+}\mathsf{W}]$ for $\Delta = 2\mathsf{W}m$. From (4.71), the inverse transform with $T = \frac{1}{2\mathsf{W}}$ satisfies

$$\begin{aligned} v_m(t) &= \sum_k v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\, e^{2\pi i(\frac{m}{T})(t-kT)} \\ &= \sum_k v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\, e^{2\pi i m t/T}. \end{aligned} \qquad (4.72)$$

Combining all of these frequency segments,

$$u(t) = \text{l.i.m.} \sum_m v_m(t) = \text{l.i.m.} \sum_{m,k} v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\, e^{2\pi i m t/T}. \qquad (4.73)$$

This converges in $\mathcal{L}_2$, but does not not necessarily converge pointwise because of the infinite summation over m. It expresses an arbitrary $\mathcal{L}_2$ function $u(t)$ in terms of the samples of each frequency slice, $v_m(t)$, of $u(t)$.

This is an orthogonal expansion in the doubly indexed set of functions

$$\{\psi_{m,k}(t) = \text{sinc}(\frac{t}{T} - k) e^{2\pi i m t/T};\ m, k \in \mathbb{Z}\}. \qquad (4.74)$$

These are the time and frequency shifts of the basic function $\psi_{0,0}(t) = \text{sinc}(\frac{t}{T})$. The time shifts are in multiples of T and the frequency shifts are in multiples of $1/T$. This set of orthogonal functions is called the set of *T-spaced sinc-weighted sinusoids.*

The T-spaced sinc-weighted sinusoids and the T-spaced truncated sinusoids are quite similar. Each function in the first set is a time and frequency translate of $\text{sinc}(\frac{t}{T})$. Each function in the second set is a time and frequency translate of $\text{rect}(\frac{t}{T})$. Both sets are made up of functions separated by multiples of T in time and $1/T$ in frequency.

[33]The boundary points between frequency segments can be ignored, as in the case for time segments.

4.7.2 Degrees of freedom

An important rule of thumb used by communication engineers is that the class of real functions that are approximately baseband-limited to W_0 and approximately time-limited to $[-T_0/2, T_0/2]$ have about $2T_0\mathsf{W}_0$ real degrees of freedom if T_0
$W_0 >> 1$. This means that any function within that class can be specified approximately by specifying about $2T_0\mathsf{W}_0$ real numbers as coefficients in an orthogonal expansion. The same rule is valid for complex functions in terms of complex degrees of freedom.

This somewhat vague statement is difficult to state precisely, since time-limited functions cannot be frequency-limited and vice-versa. However, the concept is too important to ignore simply because of lack of precision. Thus several examples are given.

First, consider applying the sampling theorem to real (complex) functions $u(t)$ that are strictly baseband-limited to W_0. Then $u(t)$ is specified by its real (complex) samples at rate $2\mathsf{W}_0$. If the samples are nonzero only within the interval $[-T_0/2, T_0/2]$, then there are about $2T_0\mathsf{W}_0$ nonzero samples, and these specify $u(t)$ within this class. Here a precise class of functions have been specified, but *functions* that are zero outside of an interval have been replaced with functions whose *samples* are zero outside of the interval.

Second, consider complex functions $u(t)$ that are again strictly baseband-limited to W_0, but now apply the sinc-weighted sinusoid expansion with $\mathsf{W} = \mathsf{W}_0/(2n + 1)$ for some positive integer n. That is, the band $[-\mathsf{W}_0, \mathsf{W}_0]$ is split into $2n + 1$ slices and each slice is expanded in a sampling-theorem expansion. Each slice is specified by samples at rate $2\mathsf{W}$, so all slices are specified collectively by samples at an aggregate rate $2\mathsf{W}_0$ as before. If the samples are nonzero only within $[-T_0/2, T_0/2]$, then there are about[34] $2T_0\mathsf{W}_0$ nonzero complex samples that specify any $u(t)$ in this class.

If the functions in this class are further constrained to be real, then the coefficients for the central frequency slice are real and the negative slices are specified by the positive slices. Thus each real function in this class is specified by about $2T_0\mathsf{W}_0$ real numbers.

This class of functions is slightly different for each choice of n, since the detailed interpretation of what "approximately time-limited" means is changing. From a more practical perspective, however, all of these expansions express an approximately baseband-limited waveform by samples at rate $2\mathsf{W}_0$. As the overall duration T_0 of the class of waveforms increases, the initial transient due to the samples centered close to $-T_0/2$ and the final transient due to samples centered close to $T_0/2$ should become unimportant relative to the rest of the waveform.

The same conclusion can be reached for functions that are strictly time-limited to $[-T_0/2, T_0/2]$ by using the truncated sinusoid expansion with coefficients outside of $[-F_0,\ F_0]$ set to 0.

In summary, all the above expansions require roughly $2\mathsf{W}_0T_0$ numbers for the approximate specification of a waveform essentially limited to time T_0 and frequency W_0 for $T_0\mathsf{W}_0$ large.

It is possible to be more precise about the number of degrees of freedom in a given time and frequency band by looking at the prolate spheroidal waveform expansion (see the Appendix, Section 5A.3). The orthogonal waveforms in this expansion maximize the energy in the given time/frequency region in a certain sense. It is perhaps simpler and better, however, to live with the very approximate nature of the arguments based on the sinc-weighted sinusoid expansion and the truncated sinusoid expansion.

[34] Calculating this number of samples carefully yields $(2n+1)\left[1+\left\lfloor \frac{T_0\mathsf{W}_0}{2n+1}\right\rfloor\right]$.

4.7.3 Aliasing — a time domain approach

Both the truncated sinusoid and the sinc-weighted sinusoid expansions are conceptually useful for understanding waveforms that are approximately time- and bandwidth-limited, but in practice, waveforms are usually sampled, perhaps at a rate much higher than twice the nominal bandwidth, before digitally processing the waveforms. Thus it is important to understand the error involved in such sampling.

Suppose an $\mathcal{L}_2$ function $u(t)$ is sampled with T-spaced samples, $\{u(kT); k \in \mathbb{Z}\}$. Let $s(t)$ denote the approximation to $u(t)$ that results from the sampling theorem expansion,

$$s(t) = \sum_k u(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right). \tag{4.75}$$

If $u(t)$ is baseband-limited to $\mathsf{W} = 1/(2T)$, then $s(t) = u(t)$, but here it is no longer assumed that $u(t)$ is baseband limited. The expansion of $u(t)$ into individual frequency slices, repeated below from (4.73), helps in understanding the difference between $u(t)$ and $s(t)$:

$$u(t) = \mathrm{l.i.m.} \sum_{m,k} v_m(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right) e^{2\pi i m t/T}, \qquad \text{where} \tag{4.76}$$

$$v_m(t) = \int \hat{u}(f)\,\mathrm{rect}(fT - m) e^{2\pi i f t}\,df. \tag{4.77}$$

For an arbitrary $\mathcal{L}_2$ function $u(t)$, the sample points $u(kT)$ might be at points of discontinuity and thus be questionable. Also (4.75) need not converge, and (4.76) might not converge pointwise. To avoid these problems, $\hat{u}(f)$ will later be restricted beyond simply being $\mathcal{L}_2$. First, however, questions of convergence are disregarded and the relevant equations are derived without questioning when they are correct.

From (4.75), the samples of $s(t)$ are given by $s(kT) = u(kT)$, and combining with (4.76),

$$s(kT) = u(kT) = \sum_m v_m(kT). \tag{4.78}$$

Thus the samples from different frequency slices get summed together in the samples of $u(t)$. This phenomenon is called *aliasing*. There is no way to tell, from the samples $\{u(kT); k \in \mathbb{Z}\}$ alone, how much contribution comes from each frequency slice and thus, as far as the samples are concerned, every frequency band is an 'alias' for every other.

Although $u(t)$ and $s(t)$ agree at the sample times, they differ elsewhere (assuming that $u(t)$ is not strictly baseband-limited to $1/(2T)$). Combining (4.78) and (4.75),

$$s(t) = \sum_k \sum_m v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k). \tag{4.79}$$

The expresssions in (4.79) and (4.76) agree at $m = 0$, so the difference between $u(t)$ and $s(t)$ is

$$u(t) - s(t) = \sum_k \sum_{m \neq 0} -v_m(kT)\mathrm{sinc}\left(\frac{t}{T} - k\right) + \sum_k \sum_{m \neq 0} v_m(kT) e^{2\pi i m t/T}\mathrm{sinc}\left(\frac{t}{T} - k\right).$$

The first term above is $v_0(t) - s(t)$, *i.e.*, the difference in the nominal baseband $[-\mathsf{W}, \mathsf{W}]$. This is the error caused by the aliased terms in $s(t)$. The second term is the energy in the nonbaseband portion of $u(t)$, which is orthogonal to the first error term. Since each term is an orthogonal expansion in the sinc-weighted sinusoids of (4.74), the energy in the error is given by[35]

$$\int \left|u(t) - s(t)\right|^2 dt = T\sum_k \Big|\sum_{m\neq 0} v_m(kT)\Big|^2 + T\sum_k\sum_{m\neq 0}\Big|v_m(kT)\Big|^2. \tag{4.80}$$

Later, when the source waveform $u(t)$ is viewed as a sample function of a random process $U(t)$, it will be seen that under reasonable conditions the expected value of these two error terms are approximately equal. Thus, if $u(t)$ is filtered by an ideal low-pass filter before sampling, then $s(t)$ becomes equal to $v_0(t)$ and only the second error term in (4.80) remains; this reduces the expected mean-squared error roughly by a factor of 2. It is often easier, however, to simply sample a little faster.

4.7.4 Aliasing — a frequency domain approach

Aliasing can be, and usually is, analyzed from a frequency domain standpoint. From (4.79), $s(t)$ can be separated into the contribution from each frequency band as

$$s(t) = \sum_m s_m(t), \qquad \text{where} \quad s_m(t) = \sum_k v_m(kT)\text{sinc}\left(\frac{t}{T} - k\right). \tag{4.81}$$

Comparing $s_m(t)$ to $v_m(t) = \sum_k v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\,e^{2\pi imt/T}$, it is seen that

$$v_m(t) = s_m(t)e^{2\pi imt/T}.$$

From the Fourier frequency shift relation, $\hat{v}_m(f) = \hat{s}_m(f - \frac{m}{T})$, so

$$\hat{s}_m(f) = \hat{v}_m(f + \frac{m}{T}). \tag{4.82}$$

Finally, since $\hat{v}_m(f) = \hat{u}(f)\,\text{rect}(fT - m)$, one sees that $\hat{v}_m(f + \frac{m}{T}) = \hat{u}(f + \frac{m}{T})\,\text{rect}(fT)$. Thus, summing (4.82) over m,

$$\hat{s}(f) = \sum_m \hat{u}(f + \frac{m}{T})\,\text{rect}[fT]. \tag{4.83}$$

Each frequency slice $\hat{v}_m(f)$ is shifted down to baseband in this equation, and then all these shifted frequency slices are summed together, as illustrated in Figure 4.10. This establishes the essence of the following aliasing theorem, which is proved in Section 5A.2.

Theorem 4.7.1 (Aliasing theorem). *Let $\hat{u}(f)$ be $\mathcal{L}_2$, and let $\hat{u}(f)$ satisfy the condition $\lim_{|f|\to\infty} \hat{u}(f)|f|^{1+\varepsilon} = 0$ for some $\varepsilon > 0$. Then $\hat{u}(f)$ is $\mathcal{L}_1$, and the inverse Fourier transform $u(t) = \int \hat{u}(f)e^{2\pi ift}\,df$ converges pointwise to a continuous bounded function. For any given $T > 0$, the sampling approximation $\sum_k u(kT)\,\text{sinc}(\frac{t}{T} - k)$ converges pointwise to a continuous bounded $\mathcal{L}_2$ function $s(t)$. The Fourier transform of $s(t)$ satisfies*

$$\hat{s}(f) = \text{l.i.m.}\sum_m \hat{u}(f + \frac{m}{T})\,\text{rect}[fT]. \tag{4.84}$$

[35] As shown by example in Exercise 4.38, $s(t)$ need not be $\mathcal{L}_2$ unless the additional restrictions of Theorem 4.7.1 are applied to $\hat{u}(f)$. In these bizarre situations, the first sum in (4.80) is infinite and $s(t)$ is a complete failure as an approximation to $u(t)$.

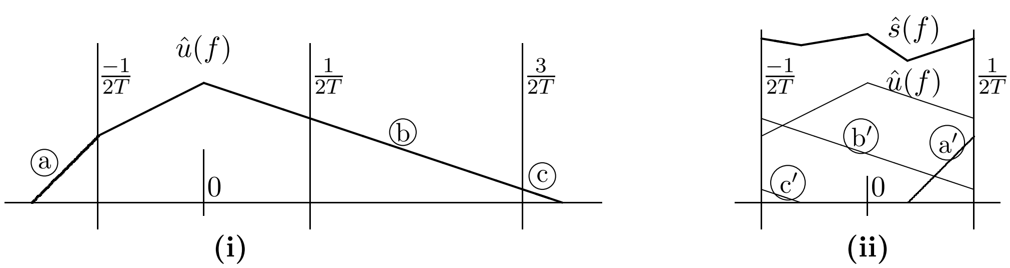


Figure 4.10: The transform $\hat{s}(f)$ of the baseband-sampled approximation $s(t)$ to $u(t)$ is constructed by folding the transform $\hat{u}(f)$ into $[-1/(2T), 1/(2T)]$. For example, using real functions for pictorial clarity, the component a is mapped into a', b into b' and c into c'. These folded components are added to obtain $\hat{s}(f)$. If $\hat{u}(f)$ is complex, then both the real and imaginary parts of $\hat{u}(f)$ must be folded in this way to get the real and imaginary parts respectively of $\hat{s}(f)$. The figure further clarifies the two terms on the right of (4.80). The first term is the energy of $\hat{u}(f) - \hat{s}(f)$ caused by the folded components in part (ii) . The final term is the energy in part (i) outside of $[-\mathcal{T}/2, T/2]$.

The condition that $\lim \hat{u}(f)f^{1+\varepsilon} = 0$ implies that $\hat{u}(f)$ goes to 0 with increasing f at a faster rate than $1/f$. Exercise 4.37 gives an example in which the theorem fails in the absence of this condition.

Without the mathematical convergence details, what the aliasing theorem says is that, corresponding to a Fourier transform pair $u(t) \leftrightarrow \hat{u}(f)$, there is another Fourier transform pair $s(t)$ and $\hat{s}(f)$; $s(t)$ is a baseband sampling expansion using the T-spaced samples of $u(t)$ and $\hat{s}(f)$ is the result of folding the transform $\hat{u}(f)$ into the band $[-\mathsf{W}, \mathsf{W}]$ with $\mathsf{W} = 1/(2T)$.

4.8 Summary

The theory of $\mathcal{L}_2$ (finite-energy) functions has been developed in this chapter. These are in many ways the ideal waveforms to study, both because of the simplicity and generality of their mathematical properties and because of their appropriateness for modeling both source waveforms and channel waveforms.

For encoding source waveforms, the general approach is

- expand the waveform into an orthogonal expansion
- quantize the coefficients in that expansion
- use discrete source coding on the quantizer output.

The distortion, measured as the energy in the difference between the source waveform and the reconstructed waveform, is proportional to the squared quantization error in the quantized coefficients.

For encoding waveforms to be transmitted over communication channels, the approach is

- map the incoming sequence of binary digits into a sequence of real or complex symbols
- use the symbols as coefficients in an orthogonal expansion.

Orthogonal expansions have been discussed in this chapter and will be further discussed in Chapter 5. Chapter 6 will discuss the choice of symbol set, the mapping from binary digits, and the choice of orthogonal expansion.

This chapter showed that every $\mathcal{L}_2$ time-limited waveform has a Fourier series, where each Fourier coefficient is given as a Lebesgue integral and the Fourier series converges in $\mathcal{L}_2$, *i.e.*, as more and more Fourier terms are used in approximating the function, the energy difference between the waveform and the approximation gets smaller and approaches 0 in the limit.

Also, by the Plancherel theorem, every $\mathcal{L}_2$ waveform $u(t)$ (time-limited or not) has a Fourier integral $\hat{u}(f)$. For each truncated approximation, $u_A(t) = u(t)\text{rect}(\frac{t}{2A})$, the Fourier integral $\hat{u}_A(f)$ exists with pointwise convergence and is continuous. The Fourier integral $\hat{u}(f)$ is then the $\mathcal{L}_2$ limit of these approximation waveforms. The inverse transform exists in the same way.

These powerful $\mathcal{L}_2$ convergence results for Fourier series and integrals are not needed for computing the Fourier transforms and series for the conventional waveforms appearing in exercises. They become important both when the waveforms are sample functions of random processes and when one wants to find limits on possible performance. In both of these situations, one is dealing with a large class of potential waveforms, rather than a single waveform, and these general results become important.

The DTFT is the frequency/time dual of the Fourier series, and the sampling theorem is simply the Fourier transform of the DTFT, combined with a little care about convergence.

The T-spaced truncated sinusoid expansion and the T-spaced sinc-weighted sinusoid expansion are two orthogonal expansions of an arbitrary $\mathcal{L}_2$ waveform. The first is formed by segmenting the waveform into T-length segments and expanding each segment in a Fourier series. The second is formed by segmenting the waveform in frequency and sampling each frequency band. The orthogonal waveforms in each are the time/frequency translates of $\text{rect}(t/T)$ for the first case and $\text{sinc}(t/T)$ for the second. Each expansion leads to the notion that waveforms roughly limited to a time interval T_0 and a baseband frequency interval F_0 have approximately $2T_0F_0$ degrees of freedom when T_0F_0 is large.

Aliasing is the ambiguity in a waveform that is represented by its T-spaced samples. If an $\mathcal{L}_2$ waveform is baseband-limited to $1/(2T)$, then its samples specify the waveform, but if the waveform has components in other bands, these components are aliased with the baseband components in the samples. The aliasing theorem says that the Fourier transform of the baseband reconstruction from the samples is equal to the original Fourier transform folded into that baseband.

4A Appendix: Supplementary material and proofs

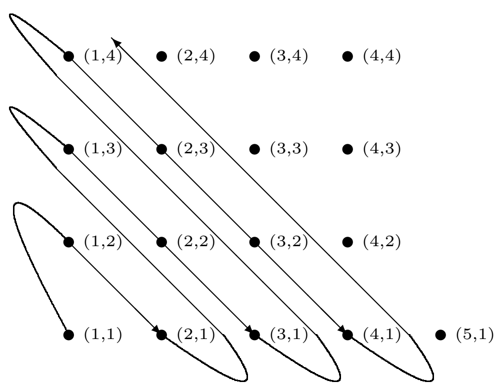
The first part of the appendix is an introduction to countable sets. These results are used throughout the chapter, and the material here can serve either as a first exposure or a review. The following three parts of the appendix provide added insight and proofs about the results on measurable sets.

4A.1 Countable sets

A collection of distinguishable objects is *countably infinite* if the objects can be put into one-to-one correspondence with the positive integers. Stated more intuitively, the collection is countably infinite if the set of elements can be arranged as a sequence $a_1, a_2, \dots$,. A set is countable if it contains either a finite or countably infinite set of elements.

Example 4A.1 (The set of all integers). The integers can be arranged as the sequence 0, -1, +1, -2, +2, -3, ... , and thus the set is countably infinite. Note that each integer appears once and only once in this sequence, and the one-to-one correspondence is $(0 \leftrightarrow 1)$, $(-1 \leftrightarrow 2)$, $(+1 \leftrightarrow 3)$, $(-2 \leftrightarrow 4)$, etc. There are many other ways to list the integers as a sequence, such as 0, -1, +1, +2, -2, +3, +4, -3, +5, ... , but, for example, listing all the non-negative integers first followed by all the negative integers is not a valid one-to-one correspondence since there are no positive integers left over for the negative integers to map into.

Example 4A.2 (The set of 2-tuples of positive integers). Figure 4.11 shows that this set is countably infinite by showing one way to list the elements in a sequence. Note that every 2-tuple is eventually reached in this list. In a weird sense, this means that there are as many positive integers as there are pairs of positive integers, but what is happening is that the integers in the 2-tuple advance much more slowly than the position in the list. For example, it can be verified that (n, n) appears in position $2n(n-1)+1$ of the list.



$1 \leftrightarrow (1,1)$
$2 \leftrightarrow (1,2)$
$3 \leftrightarrow (2,1)$
$4 \leftrightarrow (1,3)$
$5 \leftrightarrow (2,2)$
$6 \leftrightarrow (3,1)$
$7 \leftrightarrow (1,4)$
and so forth

Figure 4.11: A one-to-one correspondence between positive integers and 2-tuples of positive integers.

By combining the ideas in the previous two examples, it can be seen that the collection of all integer 2-tuples is countably infinite. With a little more ingenuity, it can be seen that the set of integer n-tuples is countably infinite for all positive integer n. Finally, it is straightforward to verify that any subset of a countable set is also countable. Also a finite union of countable sets is countable, and in fact a countable union of countable sets must be countable.

Example 4A.3. (The set of rational numbers] Each rational number can be represented by an integer numerator and denominator, and can be uniquely represented by its irreducible numerator and denominator. Thus the rational numbers can be put into one-to-one correspondence with a subset of the collection of 2-tuples of integers, and are thus countable. The rational numbers in the interval $[-T/2, T/2]$ for any given $T > 0$ form a subset of all rational numbers, and therefore are countable also.

As seen in Subsection 4.3.1, any countable set of numbers $a_1, a_2, \cdots$ can be expressed as a disjoint countable union of zero-measure sets, $[a_1, a_1], [a_2, a_2], \cdots$ so the measure of any countable set is zero. Consider a function that has the value 1 at each rational argument and 0 elsewhere.

The Lebesgue integral of that function is 0. Since rational numbers exist in every positive-sized interval of the real line, no matter how small, the Riemann integral of this function is undefined. This function is not of great practical interest, but provides insight into why Lebesgue integration is so general.

Example 4A.4 (The set of binary sequences). An example of an *uncountable* set of elements is the set of (unending) sequences of binary digits. It will be shown that this set contains uncountably many elements by assuming the contrary and constructing a contradiction. Thus, suppose we can list all binary sequences, $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \dots$. Each sequence, $\boldsymbol{a}_n$, can be expressed as $\boldsymbol{a}_n = (a_{n,1}, a_{n,2}, \dots)$, resulting in a doubly infinite array of binary digits. We now construct a new binary sequence $\boldsymbol{b} = b_1, b_2, \dots$, in the following way. For each integer $n > 0$, choose $b_n \neq a_{n,n}$; since b_n is binary, this specifies b_n for each n and thus specifies $\boldsymbol{b}$. Now $\boldsymbol{b}$ differs from each of the listed sequences in at least one binary digit, so that $\boldsymbol{b}$ is a binary sequence not on the list. This is a contradiction, since by assumption the list contains each binary sequence.

This example clearly extends to ternary sequences and sequences from any alphabet with more than one member.

Example 4A.5 (The set of real numbers in $[0, 1)$). This is another uncountable set, and the proof is very similar to that of the last example. Any real number $r \in [0, 1)$ can be represented as a binary expansion $0.r_1 r_2, \cdots$ whose elements r_k are chosen to satisfy $r = \sum_{k=1}^{\infty} r_k 2^{-k}$ and where each $r_k \in \{0, 1\}$. For example, 1/2 can be represented as 0.1, 3/8 as 0.011, etc. This expansion is unique except in the special cases where r can be represented by a finite binary expansion, $r = \sum_{k=1}^{m} r_k$; for example, 1/2 can also be represented as $0.0111 \cdots$. By convention, for each such r (other than $r = 0$) choose m as small as possible; thus in the infinite expansion, $r_m = 1$ and $r_k = 0$ for all $k > m$. Each such number can be alternatively represented with $r_m = 0$ and $r_k = 1$ for all $k > m$.

By convention, map each such r into the expansion terminating with an infinite sequence of zeros. The set of binary sequences is then the union of the representations of the reals in $[0, 1)$ and the set of binary sequences terminating in an infinite sequence of 1's. This latter set is countable because it is in one-to-one correspondence with the rational numbers of the form $\sum_{k=1}^{m} r_k 2^{-k}$ with binary r_k and finite m. Thus if the reals were countable, their union with this latter set would be countable, contrary to the known uncountability of the binary sequences.

By scaling the interval [0,1), it can be seen that the set of real numbers in any interval of non-zero size is uncountably infinite. Since the set of rational numbers in such an interval is countable, the irrational numbers must be uncountable (otherwise the union of rational and irrational numbers, *i.e.*, the reals, would be countable).

The set of irrationals in $[-T/2, T/2]$ is the complement of the rationals and thus has measure T. Each pair of distinct irrationals is separated by rational numbers. Thus the irrationals can be represented as a union of intervals only by using an uncountable union[36] of intervals, each containing a single element. The class of uncountable unions of intervals is not very interesting since it includes all subsets of $\mathbb{R}$.

[36] This might be a shock to one's intuition. Each partial union $\bigcup_{j=1}^{k} [a_j, a_j]$ of rationals has a complement which is the union of $k + 1$ intervals of non-zero width; each unit increase in k simply causes one interval in the complement to split into two smaller intervals (although maintaining the measure at T). In the limit, however, this becomes an uncountable set of separated points.

4A.2 Finite unions of intervals over $[-T/2, T/2]$

Let $\mathcal{M}_f$ be the class of finite unions of intervals, *i.e.*, the class of sets whose elements can each be expressed as $\mathcal{E} = \bigcup_{j=1}^{\ell} I_j$ where $\{I_1, \dots, I_\ell\}$ are intervals and $\ell \geq 1$ is an integer. Exercise 4.5 shows that each such $\mathcal{E} \in \mathcal{M}_f$ can be uniquely expressed as a finite union of $k \leq \ell$ *separated* intervals, say $\mathcal{E} = \bigcup_{j=1}^{k} I'_j$. The measure of $\mathcal{E}$ was defined as $\mu(\mathcal{E}) = \sum_{j=1}^{k} \mu(I'_j)$. Exercise 4.7 shows that $\mu(\mathcal{E}) \leq \sum_{j=1}^{\ell} \mu(I_j)$ for the original intervals making up $\mathcal{E}$ and shows that this holds with equality whenever $I_1, \dots, I_\ell$ are disjoint.[37]

The class $\mathcal{M}_f$ is closed under the union operation, since if $\mathcal{E}_1$ and $\mathcal{E}_2$ are each finite unions of intervals, then $\mathcal{E}_1 \cup \mathcal{E}_2$ is the union of both sets of intervals. It also follows from this that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint then

$$\mu(\mathcal{E}_1 \cup \mathcal{E}_2) = \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2). \tag{4.85}$$

The class $\mathcal{M}_f$ is also closed under the intersection operation, since, if $\mathcal{E}_1 = \bigcup_j I_{1,j}$ and $\mathcal{E}_2 = \bigcup_\ell I_{2,\ell}$, then $\mathcal{E}_1 \cap \mathcal{E}_2 = \bigcup_{j,\ell}(I_{1,j} \cap I_{2,\ell})$. Finally, $\mathcal{M}_f$ is closed under complementation. In fact, as illustrated in Figure 4.5, the complement $\overline{\mathcal{E}}$ of a finite separated union of intervals $\mathcal{E}$ is simply the union of separated intervals lying between the intervals of $\mathcal{E}$. Since $\mathcal{E}$ and its complement $\overline{\mathcal{E}}$ are disjoint and fill all of $[-T/2, T/2]$, each $\mathcal{E} \in \mathcal{M}_f$ satisfies the complement property,

$$T = \mu(\mathcal{E}) + \mu(\overline{\mathcal{E}}). \tag{4.86}$$

An important generalization of (4.85) is the following: for any $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{M}_f$,

$$\mu(\mathcal{E}_1 \cup \mathcal{E}_2) + \mu(\mathcal{E}_1 \cap \mathcal{E}_2) = \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2). \tag{4.87}$$

To see this intuitively, note that each interval in $\mathcal{E}_1 \cap \mathcal{E}_2$ is counted twice on each side of (4.87), whereas each interval in only $\mathcal{E}_1$ or only $\mathcal{E}_2$ is counted once on each side. More formally, $\mathcal{E}_1 \cup \mathcal{E}_2 = \mathcal{E}_1 \cup (\mathcal{E}_2 \cap \overline{\mathcal{E}_1})$. Since this is a disjoint union, (4.85) shows that $\mu(\mathcal{E}_1 \cup \mathcal{E}_2) = \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2 \cap \overline{\mathcal{E}_1})$. Similarly, $\mu(\mathcal{E}_2) = \mu(\mathcal{E}_2 \cap \mathcal{E}_1) + \mu(\mathcal{E}_2 \cap \overline{\mathcal{E}_1})$. Combining these equations results in (4.87).

4A.3 Countable unions and outer measure over $[-T/2, T/2]$

Let $\mathcal{M}_c$ be the class of countable unions of intervals, *i.e.*, each set $\mathcal{B} \in \mathcal{M}_c$ can be expressed as $\mathcal{B} = \bigcup_j I_j$ where $\{I_1, I_2 \dots\}$ is either a finite or countably infinite collection of intervals. The class $\mathcal{M}_c$ is closed under both the union operation and the intersection operation by the same argument as used for $\mathcal{M}_f$. $\mathcal{M}_c$ is also closed under countable unions (see Exercise 4.8) but not closed under complements or countable intersections.[38]

Each $\mathcal{B} \in \mathcal{M}_c$ can be uniquely[39] expressed as a countable union of separated intervals, say $\mathcal{B} = \bigcup_j I'_j$ where $\{I'_1, I'_2, \dots\}$ are separated (see Exercise 4.6). The measure of $\mathcal{B}$ is defined as

$$\mu(\mathcal{B}) = \sum_j \mu(I'_j). \tag{4.88}$$

[37]Recall that intervals such as (0,1], (1,2] are disjoint but not separated. A set $\mathcal{E} \in \mathcal{M}_f$ has many representations as disjoint intervals but only one as separated intervals, which is why the definition refers to separated intervals.

[38]Appendix 4A.1 shows that the complement of the rationals, *i.e.*, the set of irrationals, does not belong to $\mathcal{M}_c$. The irrationals can also be viewed as the intersection of the complements of the rationals, giving an example where M_c is not closed under countable intersections.

[39]What is unique here is the *collection* of intervals, not the *particular ordering*; this does not affect the infinite sum in (4.88) (see Exercise 4.4).

As shown in Subsection 4.3.1, the right side of (4.88) always converges to a number between 0 and T. For $B = \bigcup_j I_j$ where $I_1, I_2, \dots$, are arbitrary intervals, Exercise 4.7 establishes the following union bound,

$$\mu(\mathcal{B}) \leq \sum_j \mu(I_j) \qquad \text{with equality if } I_1, I_2, \dots \text{are disjoint.} \tag{4.89}$$

The *outer measure* $\mu^{\mathrm{o}}(\mathcal{A})$ of an arbitary set $\mathcal{A}$ was defined in (4.13) as

$$\mu^{\mathrm{o}}(\mathcal{A}) = \inf_{\mathcal{B}\in\mathcal{M}_c,\, \mathcal{A}\subseteq\mathcal{B}} \mu(\mathcal{B}). \tag{4.90}$$

Note that $[-T/2, T/2]$ is a cover of $\mathcal{A}$ for all $\mathcal{A}$ (recall that only sets in $[-T/2, T/2]$ are being considered). Thus $\mu^{\mathrm{o}}(\mathcal{A})$ must lie between 0 and T for all $\mathcal{A}$. Also, for any two sets $\mathcal{A} \subseteq \mathcal{A}'$, any cover of $\mathcal{A}'$ also covers $\mathcal{A}$. This implies the *subset inequality* for outer measure,

$$\mu^{\mathrm{o}}(\mathcal{A}) \leq \mu^{\mathrm{o}}(\mathcal{A}') \qquad \text{for } \mathcal{A} \subseteq \mathcal{A}'. \tag{4.91}$$

The following lemma develops the *union bound* for outer measure. Its proof illustrates several techniques that will be used frequently.

Lemma 4A.1. *Let $\mathcal{S} = \bigcup_k \mathcal{A}_k$ be a countable union of arbitrary sets in $[-T/2, T/2]$. Then*

$$\mu^{\mathrm{o}}(\mathcal{S}) \leq \sum_k \mu^{\mathrm{o}}(\mathcal{A}_k). \tag{4.92}$$

Proof: The approach is to first establish an arbitrarily tight cover to each $\mathcal{A}_k$ and then show that the union of these covers is a cover for $\mathcal{S}$. Specifically, let ε be an arbitrarily small positive number. For each $k \geq 1$, the infimum in (4.90) implies that covers exist with measures arbitrarily little greater than that infimum. Thus a cover B_k to $\mathcal{A}_k$ exists with

$$\mu(\mathcal{B}_k) \leq \varepsilon 2^{-k} + \mu^{\mathrm{o}}(\mathcal{A}_k).$$

For each k, let $\mathcal{B}_k = \bigcup_j I'_{j,k}$ where $I'_{1,k}, I'_{2,k}, \dots$ represents $\mathcal{B}_k$ by separated intervals. Then $\mathcal{B} = \bigcup_k \mathcal{B}_k = \bigcup_k \bigcup_j I'_{j,k}$ is a countable union of intervals, so from (4.89) and Exercise 4.4,

$$\mu(\mathcal{B}) \leq \sum_k \sum_j \mu(I'_{j,k}) = \sum_k \mu(\mathcal{B}_k)$$

Since $\mathcal{B}_k$ covers $\mathcal{A}_k$ for each k, it follows that $\mathcal{B}$ covers $\mathcal{S}$. Since $\mu^{\mathrm{o}}(S)$ is the infimum of its covers,

$$\mu^{\mathrm{o}}(S) \leq \mu(B) \leq \sum_k \mu(\mathcal{B}_k) \leq \sum_k \left(\varepsilon 2^{-k} + \mu^{\mathrm{o}}(\mathcal{A}_k)\right) = \varepsilon + \sum_k \mu^{\mathrm{o}}(\mathcal{A}_k).$$

Since $\varepsilon > 0$ is arbitrary, (4.92) follows. □

An important special case is the union of any set $\mathcal{A}$ and its complement $\overline{\mathcal{A}}$. Since $[-T/2, T/2] = \mathcal{A} \cup \overline{\mathcal{A}}$,

$$T \leq \mu^{\mathrm{o}}(\mathcal{A}) + \mu^{\mathrm{o}}(\overline{\mathcal{A}}). \tag{4.93}$$

The next subsection will define measurability and measure for arbitrary sets. Before that, the following theorem shows both that countable unions of intervals are measurable and that their measure, as defined in (4.88), is consistent with the general definition to be given later.

Theorem 4A.1. *Let* $\mathcal{B} = \bigcup_j I_j$ *where* $\{I_1, I_2, \dots\}$ *is a countable collection of intervals in* $[-T/2, T/2]$ *(i.e.,* $\mathcal{B} \in \mathcal{M}_c$*). Then*

$$\mu^{\mathrm{o}}(\mathcal{B}) + \mu^{\mathrm{o}}(\overline{\mathcal{B}}) = T \qquad \text{and} \tag{4.94}$$

$$\mu^{\mathrm{o}}(\mathcal{B}) = \mu(B). \tag{4.95}$$

Proof: Let $\{I'_j; j \geq 1\}$ be the collection of separated intervals representing $\mathcal{B}$ and let

$$\mathcal{E}^k = \bigcup\nolimits_{j=1}^{k} I'_j; \qquad \text{then}$$

$$\mu(\mathcal{E}^1) \leq \mu(\mathcal{E}^2) \leq \mu(\mathcal{E}^3) \leq \cdots \leq \lim_{k\to\infty} \mu(\mathcal{E}^k) = \mu(\mathcal{B}).$$

For any $\varepsilon > 0$, choose k large enough that

$$\mu(\mathcal{E}^k) \geq \mu(B) - \varepsilon. \tag{4.96}$$

The idea of the proof is to approximate $\mathcal{B}$ by $\mathcal{E}^k$, which, being in $\mathcal{M}_f$, satisfies $T = \mu(\mathcal{E}^k) + \mu(\overline{\mathcal{E}^k})$. Thus,

$$\mu(\mathcal{B}) \leq \mu(\mathcal{E}^k) + \varepsilon = T - \mu(\overline{\mathcal{E}^k}) + \varepsilon \leq T - \mu^{\mathrm{o}}(\overline{\mathcal{B}}) + \varepsilon, \tag{4.97}$$

where the final inequality follows because $\mathcal{E}^k \subseteq \mathcal{B}$ and thus $\overline{\mathcal{B}} \subseteq \overline{\mathcal{E}^k}$ and $\mu^{\mathrm{o}}(\overline{\mathcal{B}}) \leq \mu(\overline{\mathcal{E}^k})$.

Next, since $\mathcal{B} \in \mathcal{M}_c$ and $\mathcal{B} \subseteq \mathcal{B}$, $\mathcal{B}$ is a cover of itself and is a choice in the infimum defining $\mu^{\mathrm{o}}(B)$; thus $\mu^{\mathrm{o}}(\mathcal{B}) \leq \mu(\mathcal{B})$. Combining this with (4.97), $\mu^{\mathrm{o}}(\mathcal{B}) + \mu^{\mathrm{o}}(\overline{\mathcal{B}}) \leq T + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies

$$\mu^{\mathrm{o}}(\mathcal{B}) + \mu^{\mathrm{o}}(\overline{\mathcal{B}}) \leq T. \tag{4.98}$$

This combined with (4.93) establishes (4.94). Finally, substituting $T \leq \mu^{\mathrm{o}}(\mathcal{B}) + \mu^{\mathrm{o}}(\overline{\mathcal{B}})$ into (4.97), $\mu(\mathcal{B}) \leq \mu^{\mathrm{o}}(\mathcal{B}) + \varepsilon$. Since $\mu^{\mathrm{o}}(\mathcal{B}) \leq \mu(\mathcal{B})$ and $\varepsilon > 0$ is arbitrary, this establishes (4.95). □

Finally, before proceeding to arbitrary measurable sets, the joint union and intersection property, (4.87), is extended to $\mathcal{M}_c$.

Lemma 4A.2. *Let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be arbitrary sets in* $\mathcal{M}_c$*. Then*

$$\mu(\mathcal{B}_1 \cup \mathcal{B}_2) + \mu(\mathcal{B}_1 \cap \mathcal{B}_2) = \mu(\mathcal{B}_1) + \mu(\mathcal{B}_2). \tag{4.99}$$

Proof: Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be represented respectively by separated intervals, $\mathcal{B}_1 = \bigcup_j I_{1,j}$ and $\mathcal{B}_2 = \bigcup_j I_{2,j}$. For $\ell = 1, 2$, let $\mathcal{E}_\ell^k = \bigcup_{j=1}^{k} I_{\ell,j}$ and $\mathcal{D}_\ell^k = \bigcup_{j=k+1}^{\infty} I_{\ell,j}$. Thus $\mathcal{B}_\ell = \mathcal{E}_\ell^k \cup \mathcal{D}_\ell^k$ for each integer $k \geq 1$ and $\ell = 1, 2$. The proof is based on using $\mathcal{E}_\ell^k$, which is in $\mathcal{M}_f$ and satisfies the joint union and intersection property, as an approximation to $\mathcal{B}_\ell$. To see how this goes, note that

$$\mathcal{B}_1 \cap \mathcal{B}_2 = (\mathcal{E}_1^k \cup \mathcal{D}_1^k) \cap (\mathcal{E}_2^k \cup \mathcal{D}_2^k) = (\mathcal{E}_1^k \cap \mathcal{E}_2^k) \cup (\mathcal{E}_1^k \cap \mathcal{D}_2^k) \cup (\mathcal{D}_1^k \cap \mathcal{B}_2).$$

For any $\varepsilon > 0$ we can choose k large enough that $\mu(\mathcal{E}_\ell^k) \geq \mu(\mathcal{B}_\ell) - \varepsilon$ and $\mu(\mathcal{D}_\ell^k) \leq \varepsilon$ for $\ell = 1, 2$. Using the subset inequality and the union bound, we then have

$$\begin{aligned}\mu(\mathcal{B}_1 \cap \mathcal{B}_2) &\leq \mu(\mathcal{E}_1^k \cap \mathcal{E}_2^k) + \mu(\mathcal{D}_2^k) + \mu(\mathcal{D}_1^k)\\ &\leq \mu(\mathcal{E}_1^k \cap \mathcal{E}_2^k) + 2\varepsilon.\end{aligned}$$

By a similar but simpler argument,

$$\begin{aligned}\mu(\mathcal{B}_1 \cup \mathcal{B}_2) &\leq \mu(\mathcal{E}_1^k \cup \mathcal{E}_2^k) + \mu(\mathcal{D}_1^k) + \mu(\mathcal{D}_2^k)\\ &\leq \mu(\mathcal{E}_1^k \cup \mathcal{E}_2^k) + 2\varepsilon.\end{aligned}$$

Combining these inequalities and using (4.87) on $\mathcal{E}_1^k \subseteq \mathcal{M}_f$ and $\mathcal{E}_2^k \subseteq \mathcal{M}_f$, we have

$$\begin{aligned}\mu(\mathcal{B}_1 \cap \mathcal{B}_2) + \mu(\mathcal{B}_1 \cup \mathcal{B}_2) &\leq \mu(\mathcal{E}_1^k \cap \mathcal{E}_2^k) + \mu(\mathcal{E}_1^k \cup \mathcal{E}_2^k) + 4\varepsilon\\ &= \mu(\mathcal{E}_1^k) + \mu(\mathcal{E}_2^k) + 4\varepsilon\\ &\leq \mu(\mathcal{B}_1) + \mu(\mathcal{B}_2) + 4\varepsilon.\end{aligned}$$

where we have used the subset inequality in the final inequality.

For a bound in the opposite direction, we start with the subset inequality,

$$\begin{aligned}\mu(\mathcal{B}_1 \cup \mathcal{B}_2) + \mu(\mathcal{B}_1 \cap \mathcal{B}_2) &\geq \mu(\mathcal{E}_1^k \cup \mathcal{E}_2^k) + \mu(\mathcal{E}_1^k \cap \mathcal{E}_2^k)\\ &= \mu(\mathcal{E}_1^k) + \mu(\mathcal{E}_2^k)\\ &\geq \mu(B_1) + \mu(\mathcal{B}_2) - 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, these two bounds establish (4.99). □

4A.4 Arbitrary measurable sets over $[-T/2, T/2]$

An arbitrary set $\mathcal{A} \in [-T/2, T/2]$ was defined to be *measurable* if

$$T = \mu^{\mathrm{o}}(\mathcal{A}) + \mu^{\mathrm{o}}(\overline{\mathcal{A}}). \tag{4.100}$$

The *measure* of a measurable set was defined to be $\mu(\mathcal{A}) = \mu^{\mathrm{o}}(\mathcal{A})$. The class of measurable sets is denoted as $\mathcal{M}$. Theorem 4A.1 shows that each set $\mathcal{B} \in \mathcal{M}_c$ is measurable, *i.e.*, $\mathcal{B} \in \mathcal{M}$ and thus $\mathcal{M}_f \subseteq \mathcal{M}_c \subseteq \mathcal{M}$. The measure of $\mathcal{B} \in \mathcal{M}_c$ is $\mu(\mathcal{B}) = \sum_j \mu(I_j)$ for any disjoint sequence of intervals, $I_1, I_2, \dots$, whose union is $\mathcal{B}$.

Although the complements of sets in $\mathcal{M}_c$ are not necessarily in $\mathcal{M}_c$ (as seen from the rational number example), they must be in $\mathcal{M}$; in fact, from (4.100), all sets in $\mathcal{M}$ have complements in $\mathcal{M}$, *i.e.*, M is closed under complements. We next show that $\mathcal{M}$ is closed under, first, finite, and then, countable, unions and intersections. The key to these results is to first show that the joint union and intersection property is valid for outer measure.

Lemma 4A.3. *For any measurable sets $\mathcal{A}_1$ and $\mathcal{A}_2$,*

$$\mu^{\mathrm{o}}(\mathcal{A}_1 \cup \mathcal{A}_2) + \mu^{\mathrm{o}}(\mathcal{A}_1 \cap \mathcal{A}_2) = \mu^{\mathrm{o}}(A_1) + \mu^{\mathrm{o}}(A_2). \tag{4.101}$$

Proof: The proof is very similar to that of lemma 4A.2, but here we use sets in $\mathcal{M}_c$ to approximate those in $\mathcal{M}$. For any $\varepsilon > 0$, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be covers of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively such that

$\mu(\mathcal{B}_\ell) \leq \mu^{\rm o}(\mathcal{A}_\ell) + \varepsilon$ for $\ell = 1, 2$. Let $\mathcal{D}_\ell = \mathcal{B}_\ell \cap \overline{\mathcal{A}}_\ell$ for $\ell = 1, 2$. Note that $\mathcal{A}_\ell$ and $\mathcal{D}_\ell$ are disjoint and $\mathcal{B}_\ell = \mathcal{A}_\ell \cup \mathcal{D}_\ell$.

$$\mathcal{B}_1 \cap \mathcal{B}_2 = (\mathcal{A}_1 \cup \mathcal{D}_1) \cap (\mathcal{A}_2 \cup \mathcal{D}_2) = (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{D}_1 \cap \mathcal{A}_2) \cup (\mathcal{B}_1 \cap \mathcal{D}_2).$$

Using the union bound and subset inequality for outer measure on this and the corresponding expansion of $\mathcal{B}_1 \cup \mathcal{B}_2$, we get

$$\begin{aligned} \mu(\mathcal{B}_1 \cap \mathcal{B}_2) &\leq \mu^{\rm o}(\mathcal{A}_1 \cap \mathcal{A}_2) + \mu^{\rm o}(\mathcal{D}_1) + \mu^{\rm o}(D_2) \leq \mu^{\rm o}(\mathcal{A}_1 \cap \mathcal{A}_2) + 2\varepsilon \\ \mu(\mathcal{B}_1 \cup \mathcal{B}_2) &\leq \mu^{\rm o}(\mathcal{A}_1 \cup \mathcal{A}_2) + \mu^{\rm o}(D_1) + \mu^{\rm o}(\mathcal{D}_2) \leq \mu^{\rm o}(\mathcal{A}_1 \cup \mathcal{A}_2) + 2\varepsilon, \end{aligned}$$

where we have also used the fact (see Exercise 4.9) that $\mu^{\rm o}(\mathcal{D}_\ell) \leq \varepsilon$ for $\ell = 1, 2$. Summing these inequalities and rearranging terms,

$$\begin{aligned} \mu^{\rm o}(\mathcal{A}_1 \cup \mathcal{A}_2) + \mu^{\rm o}(\mathcal{A}_1 \cap \mathcal{A}_2) &\geq \mu(\mathcal{B}_1 \cap \mathcal{B}_2) + \mu(\mathcal{B}_1 \cup \mathcal{B}_2) - 4\varepsilon \\ &= \mu(\mathcal{B}_1) + \mu(\mathcal{B}_2) - 4\varepsilon \\ &\geq \mu^{\rm o}(\mathcal{A}_1) + \mu^{\rm o}(\mathcal{A}_2) - 4\varepsilon, \end{aligned}$$

where we have used (4.99) and then used $\mathcal{A}_\ell \subseteq \mathcal{B}_\ell$ for $\ell = 1, 2$. Using the subset inequality and (4.99) to bound in the opposite direction,

$$\mu(\mathcal{B}_1) + \mu(\mathcal{B}_2) = \mu(\mathcal{B}_1 \cup \mathcal{B}_2) + \mu(\mathcal{B}_1 \cap \mathcal{B}_2) \geq \mu^{\rm o}(\mathcal{A}_1 \cup \mathcal{A}_2) + \mu^{\rm o}(\mathcal{A}_1 \cap \mathcal{A}_2).$$

Rearranging and using $\mu(\mathcal{B}_\ell) \leq \mu^{\rm o}(\mathcal{A}_\ell) + \varepsilon$,

$$\mu^{\rm o}(\mathcal{A}_1 \cup \mathcal{A}_2) + \mu^{\rm o}(\mathcal{A}_1 \cap \mathcal{A}_2) \leq \mu^{\rm o}(\mathcal{A}_1) + \mu^{\rm o}(\mathcal{A}_2) + 2\varepsilon.$$

Siince ε is arbitrary, these bounds establish (4.101). □

Theorem 4A.2. *Assume $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{M}$. Then $\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{M}$ and $\mathcal{A}_1 \cap \mathcal{A}_2 \in \mathcal{M}$.*

Proof: Apply (4.101) to $\overline{\mathcal{A}_1}$ and $\overline{\mathcal{A}_2}$, getting

$$\mu^{\rm o}(\overline{\mathcal{A}_1} \cup \overline{\mathcal{A}_2}) + \mu^{\rm o}(\overline{\mathcal{A}_1} \cap \overline{\mathcal{A}_2}) = \mu^{\rm o}(\overline{\mathcal{A}_1}) + \mu^{\rm o}(\overline{\mathcal{A}_2}).$$

Rewriting $\overline{\mathcal{A}_1} \cup \overline{\mathcal{A}_2}$ as $\overline{\mathcal{A}_1 \cap \mathcal{A}_2}$ and $\overline{\mathcal{A}_1} \cap \overline{\mathcal{A}_2}$ by $\overline{\mathcal{A}_1 \cup \mathcal{A}_2}$ and adding this to (4.101),

$$\begin{aligned} &\left[\mu^{\rm o}(\mathcal{A}_1 \cup \mathcal{A}_2) + \mu^{\rm o}(\overline{\mathcal{A}_1 \cup \mathcal{A}_2}\right] + \left[\mu^{\rm o}(\mathcal{A}_1 \cap \mathcal{A}_2) + \mu^{\rm o}(\overline{\mathcal{A}_1 \cap \mathcal{A}_2})\right] \\ &\quad = \mu^{\rm o}(\mathcal{A}_1) + \mu^{\rm o}(\mathcal{A}_2) + \mu^{\rm o}(\overline{\mathcal{A}_1}) + \mu^{\rm o}(\overline{\mathcal{A}_2}) = 2T, \end{aligned} \tag{4.102}$$

where we have used (4.100). Each of the bracketed terms above is at least T from (4.93), so each term must be exactly T. Thus $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2$ are measurable. □

Since $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2$ are measurable if $\mathcal{A}_1$ and $\mathcal{A}_2$ are, the joint union and intersection property holds for measure as well as outer measure for all measurable functions, *i.e.*,

$$\mu(\mathcal{A}_1 \cup \mathcal{A}_2) + \mu(\mathcal{A}_1 \cap \mathcal{A}_2) = \mu(A_1) + \mu(A_2). \tag{4.103}$$

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint, then (4.103) simplifies to the additivity property

$$\mu(\mathcal{A}_1 \cup \mathcal{A}_2) = \mu(A_1) + \mu(\mathcal{A}_2). \tag{4.104}$$

Actually, (4.103) shows that (4.104) holds whenever $\mu(\mathcal{A}_1 \cap \ A_2) = 0$. That is, $\mathcal{A}_1$ and $\mathcal{A}_2$ need not be disjoint, but need only have an intersection of zero measure. This is another example in which sets of zero measure can be ignored.

The following theorem shows that $\mathcal{M}$ is closed over disjoint countable unions and that $\mathcal{M}$ is countably additive.

Theorem 4A.3. *Assume that $\mathcal{A}_j \in \mathcal{M}$ for each integer $j \geq 1$ and that $\mu(\mathcal{A}_j \cap \mathcal{A}_\ell) = 0$ for all $j \neq \ell$. Let $\mathcal{A} = \bigcup_j \mathcal{A}_j$. Then $\mathcal{A} \in \mathcal{M}$ and*

$$\mu(A) = \sum_j \mu(\mathcal{A}_j). \tag{4.105}$$

Proof: Let $\mathcal{A}^k = \bigcup_{j=1}^k \mathcal{A}_j$ for each integer $k \geq 1$. Then $\mathcal{A}^{k+1} = \mathcal{A}^k \cup \mathcal{A}_{k+1}$ and, by induction on the previous theorem, $\mathcal{A}^k \in \mathcal{M}$. It also follows that

$$\mu(\mathcal{A}^k) = \sum_{j=1}^k \mu(A_j).$$

The sum on the right is nondecreasing in k and bounded by T, so the limit as $k \to \infty$ exists. Applying the union bound for outer measure to $\mathcal{A}$,

$$\mu^{\mathrm{o}}(\mathcal{A}) \leq \sum_j \mu^{\mathrm{o}}(\mathcal{A}_j) = \lim_{k\to\infty} \mu^{\mathrm{o}}(\mathcal{A}^k) = \lim_{k\to\infty} \mu(\mathcal{A}^k). \tag{4.106}$$

Since $\mathcal{A}^k \subseteq \mathcal{A}$, we see that $\overline{\mathcal{A}} \subseteq \overline{\mathcal{A}^k}$ and $\mu^{\mathrm{o}}(\overline{\mathcal{A}}) \leq \mu(\overline{\mathcal{A}^k}) = T - \mu(\mathcal{A}^k)$. Thus

$$\mu^{\mathrm{o}}(\overline{\mathcal{A}}) \leq T - \lim_{k\to\infty} \mu(\mathcal{A}^k). \tag{4.107}$$

Adding (4.106) and (4.107) shows that $\mu^{\mathrm{o}}(\mathcal{A}) + \mu^{\mathrm{o}}(\overline{\mathcal{A}}) \leq T$. Combining with (4.93), $\mu^{\mathrm{o}}(\mathcal{A}) + \mu^{\mathrm{o}}(\overline{\mathcal{A}}) = T$ and (4.106) and (4.107) are satisfied with equality. Thus $\mathcal{A} \in \mathcal{M}$ and countable additivity, (4.105), is satisfied. □

Next it is shown that $\mathcal{M}$ is closed under arbitrary countable unions and intersections.

Theorem 4A.4. *Assume that $\mathcal{A}_j \in \mathcal{M}$ for each integer $j \geq 1$. Then $\mathcal{A} = \bigcup_j \mathcal{A}_j$ and $\mathcal{D} = \bigcap_j \mathcal{A}_j$ are both in $\mathcal{M}$.*

Proof: Let $\mathcal{A}'_1 = \mathcal{A}_1$ and, for each $k \geq 1$, let $\mathcal{A}^k = \bigcup_{j=1}^k \mathcal{A}_j$ and let $\mathcal{A}'_{k+1} = \mathcal{A}_{k+1} \cap \overline{\mathcal{A}^k}$. By induction, the sets $\mathcal{A}'_1, \mathcal{A}'_2, \dots$, are disjoint and measurable and $\mathcal{A} = \bigcup_j \mathcal{A}'_j$. Thus, from Theorem 4A.3, $\mathcal{A}$ is measurable. Next suppose $\mathcal{D} = \cap \mathcal{A}_j$. Then $\overline{\mathcal{D}} = \cup \overline{\mathcal{A}_j}$. Thus, $\overline{\mathcal{D}} \in M$, so $\mathcal{D} \in \mathcal{M}$ also. □

Proof of Theorem 4.3.1: The first two parts of Theorem 4.3.1 are Theorems 4A.4 and 4A.3. The third part, that $\mathcal{A}$ is measurable with zero measure if $\mu^{\mathrm{o}}(\mathcal{A}) = 0$, follows from $T \leq \mu^{\mathrm{o}}(\mathcal{A}) + \mu^{\mathrm{o}}(\overline{\mathcal{A}}) = \mu^{\mathrm{o}}(\overline{\mathcal{A}})$ and $\mu^{\mathrm{o}}(\overline{\mathcal{A}}) \leq T$, *i.e.*, that $\mu^{\mathrm{o}}(\overline{\mathcal{A}}) = T$. □

Sets of zero measure are quite important in understanding Lebesgue integration, so it is important to know whether there are also uncountable sets of points that have zero measure. The answer is yes; a simple example follows.

Example 4A.6 (The Cantor set). Express each point in the interval (0,1) by a ternary expansion. Let $\mathcal{B}$ be the set of points in (0,1) for which that expansion contains only 0's and 2's and is also nonterminating. Thus $\mathcal{B}$ excludes the interval $[1/3, 2/3)$, since all these expansions start with 1. Similarly, $\mathcal{B}$ excludes $[1/9, 2/9)$ and $[7/9, 8/9)$, since the second digit is 1 in these expansions. The right end point for each of these intervals is also excluded since it has a terminating expansion. Let $\mathcal{B}_n$ be the set of points with no 1 in the first n digits of the ternary expansion. Then $\mu(\mathcal{B}_n) = (2/3)^n$. Since $\mathcal{B}$ is contained in $\mathcal{B}_n$ for each $n \geq 1$, $\mathcal{B}$ is measurable and $\mu(\mathcal{B}) = 0$.

The expansion for each point in $\mathcal{B}$ is a binary sequence (viewing 0 and 2 as the binary digits here). There are uncountably many binary sequences (see Section 4A.1), and this remains true when the countable number of terminating sequences are removed. Thus we have demonstrated an uncountably infinite set of numbers with zero measure.

Not all point sets are Lebesgue measurable, and an example follows.

Example 4A.7 (A non-measurable set). Consider the interval $[0, 1)$. We define a collection of equivalence classes where two points in $[0, 1)$ are in the same equivalence class if the difference between them is rational. Thus one equivalence class consists of the rationals in [0,1). Each other equivalence class consists of a countably infinite set of irrationals whose differences are rational. This partitions $[0, 1)$ into an uncountably infinite set of equivalence classes. Now consider a set $\mathcal{A}$ that contains exactly one number chosen from each equivalence class. We will assume that $\mathcal{A}$ is measurable and show that this leads to a contradiction.

For the given set $\mathcal{A}$, let $\mathcal{A} + r$, for r rational in $(0, 1)$, denote the set that results from mapping each $t \in \mathcal{A}$ into either $t + r$ or $t + r - 1$, whichever lies in $[0, 1)$. The set $\mathcal{A} + r$ is thus the set $\mathcal{A}$, shifted by r, and then rotated to lie in $[0, 1)$. By looking at outer measures, it is easy to see that $\mathcal{A} + r$ is measurable if $\mathcal{A}$ is and that both then have the same measure. Finally, each $t \in [0, 1)$ lies in exactly one equivalence class, and if τ is the element of $\mathcal{A}$ in that equivalence class, then t lies in $\mathcal{A}+r$ where $r = t-\tau$ or $t-\tau+1$. In other words, $[0, 1) = \bigcup_r (\mathcal{A}+r)$ and the sets $\mathcal{A}+r$ are disjoint. Assuming that $\mathcal{A}$ is measurable, Theorem 4A.3 asserts that $1 = \sum_r \mu(\mathcal{A}+r)$. The sum on the right, however, is 0 if $\mu(\mathcal{A}) = 0$ and infinite if $\mu(\mathcal{A}) > 0$, establishing the contradiction.

4.E Exercises

4.1. (Fourier series) (a) Consider the function $u(t) = \text{rect}(2t)$ of Figure 4.2. Give a general expression for the Fourier series coefficients for the Fourier series over $[-1/2, 1/2]$. and show that the series converges to 1/2 at each of the end points, -1/4 and 1/4. Hint: You don't need to know anything about convergence here.

(b) Represent the same function as a Fourier series over the interval $[-1/4, 1/4]$. What does this series converge to at -1/4 and 1/4? Note from this exercise that the Fourier series depends on the interval over which it is taken.

4.2. (Energy equation) Derive (4.6), the energy equation for Fourier series. Hint: Substitute the Fourier series for $u(t)$ into $\int u(t)u^*(t)\,dt$. Don't worry about convergence or interchange of limits here.

4.3. (Countability) As shown in Appendix 4A.1, many subsets of the real numbers, including the integers and the rationals, are countable. Sometimes, however, it is necessary to give up the ordinary numerical ordering in listing the elements of these subsets. This exercise shows that this is sometimes inevitable.

(a) Show that every listing of the integers (such as $0, -1, 1, -2, \dots$) fails to preserve the numerical ordering of the integers (hint: assume such a numerically ordered listing exists and show that it can have no first element (*i.e.*, no smallest element.)

(b) Show that the rational numbers in the interval (0, 1) cannot be listed in a way that preserves their numerical ordering.

(c) Show that the rationals in [0,1] cannot be listed with a preservation of numerical ordering (the first element is no problem, but what about the second?).

4.4. (Countable sums) Let $a_1, a_2, \dots ,$ be a countable set of non-negative numbers and assume that $s_a(k) = \sum_{j=1}^{k} a_j \leq A$ for all k and some given $A > 0$.

(a) Show that the limit $\lim_{k\to\infty} s_a(k)$ exists with some value S_a between 0 and A. (Use any level of mathematical care that you feel comfortable with.)

(b) Now let $b_1, b_2, \dots ,$ be another ordering of the numbers $a_1, a_2, \dots ,$. That is, let $b_1 = a_{j(1)}, b_2 = a_{j(2)}, \dots , b_\ell = a_{j(\ell)}, \dots ,$ where $j(\ell)$ is a permutation of the positive integers, *i.e.*, a one-to-one function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$. Let $s_b(k) = \sum_{\ell=1}^{k} b_\ell$. Show that $\lim_{k\to\infty} s_b(k) \leq S_a$. Hint: Note that

$$\sum_{\ell=1}^{k} b_\ell = \sum_{\ell=1}^{k} a_{j(\ell)}.$$

(c) Define $S_b = \lim_{k\to\infty} s_b(k)$ and show that $S_b \geq S_a$. Hint: Consider the inverse permuation, say $\ell^{-1}(j)$, which for given j' is that ℓ for which $j(\ell) = j'$. Note that you have shown that a countable sum of non-negative elements does not depend on the order of summation.

(d) Show that the above result is not necessarily true for a countable sum of numbers that can be positive or negative. Hint: consider alternating series.

4.5. (Finite unions of intervals) Let $\mathcal{E} = \bigcup_{j=1}^{\ell} I_j$ be the union of $\ell \geq 2$ arbitrary nonempty intervals. Let a_j and b_j denote the left and right end points respectively of I_j; each end point can be included or not. Assume the intervals are ordered so that $a_1 \leq a_2 \leq \cdots \leq a_\ell$.

(a) For $\ell = 2$, show that either I_1 and I_2 are separated or that $\mathcal{E}$ is a single interval whose left end point is a_1.

(b) For $\ell > 2$ and $2 \le k < \ell$, let $\mathcal{E}^k = \bigcup_{j=1}^k I_j$. Give an algorithm for constructing a union of separated intervals for $\mathcal{E}^{k+1}$ given a union of separated intervals for $\mathcal{E}^k$.

(c) Note that using part (b) inductively yields a representation of $\mathcal{E}$ as a union of separated intervals. Show that the left end point for each separated interval is drawn from $a_1, \dots, a_\ell$ and the right end point is drawn from $b_1, \dots, b_\ell$.

(d) Show that this representation is unique, *i.e.*., that $\mathcal{E}$ can not be represented as the union of any other set of separated intervals. Note that this means that $\mu(\mathcal{E})$ is defined unambiguously in (4.9).

4.6. (Countable unions of intervals) Let $\mathcal{B} = \bigcup_j I_j$ be a countable union of arbitrary (perhaps intersecting) intervals. For each $k \ge 1$, let $\mathcal{B}^k = \bigcup_{j=1}^k I_j$ and for each $k \ge j$, let $I_{j,k}$ be the separated interval in $\mathcal{B}^k$ containing I_j (see Exercise 4.5).

(a) For each $k \ge j \ge 1$, show that $I_{j,k} \subseteq I_{j,k+1}$.

(b) Let $\bigcup_{k=j}^{\infty} I_{j,k} = I'_j$. Explain why I'_j is an interval and show that $I'_j \subseteq \mathcal{B}$.

(c) For any i, j, show that either $I'_j = I'_i$ or I'_j and I'_i are separated intervals.

(d) Show that the sequence $\{I'_j; 1 \le j < \infty\}$ with repetitions removed is a countable separated-interval representation of B.

(e) Show that the collection $\{I'_j; j \ge 1\}$ with repetitions removed is unique; *i.e.*, show that if an arbitrary interval I is contained in $\mathcal{B}$, then it is contained in one of the I'_j. Note however that the ordering of the I'_j is not unique.

4.7. (Union bound for intervals) Prove the validity of the union bound for a countable collection of intervals in (4.89). The following steps are suggested:

(a) Show that if $\mathcal{B} = I_1 \bigcup I_2$ for arbitrary intervals I_1, I_2, then $\mu(\mathcal{B}) \le \mu(I_1) + \mu(I_2)$ with equality if I_1 and I_2 are disjoint. Note: this is true by definition if I_1 and I_2 are separated, so you need only treat the cases where I_1 and I_2 intersect or are disjoint but not separated.

(b) Let $\mathcal{B}^k = \bigcup_{j=1}^k I_j$ be represented as the union of say m_k separated intervals ($m_k \le k$), so $\mathcal{B}^k = \bigcup_{j=1}^{m_k} I'_j$. Show that $\mu(\mathcal{B}^k \bigcup I_{k+1}) \le \mu(\mathcal{B}^k) + \mu(I_{k+1})$ with equality if $\mathcal{B}^k$ and I_{k+1} are disjoint.

(c) Use finite induction to show that if $\mathcal{B} = \bigcup_{j=1}^k I_j$ is a finite union of arbitrary intervals, then $\mu(B) \le \sum_{j=1}^k \mu(I_j)$ with equality if the intervals are disjoint.

(d) Extend part (c) to a countably infinite union of intervals.

4.8. For each positive integer n, let $\mathcal{B}_n$ be a countable union of intervals. Show that $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is also a countable union of intervals. Hint: Look at Example 4A.2 in Section 4A.1.

4.9. (Measure and covers) Let $\mathcal{A}$ be an arbitrary measurable set in $[-T/2, T/2]$ and let $\mathcal{B}$ be a cover of A. Using only results derived prior to Lemma 4A.3, show that $\mu^{\rm o}(\mathcal{B} \cap \overline{\mathcal{A}}) = \mu(\mathcal{B}) - \mu(\mathcal{A})$. You may use the following steps if you wish.

(a) Show that $\mu^{\rm o}(B \cap \overline{A}) \ge \mu(\mathcal{B}) - \mu(A)$.

(b) For any $\delta > 0$, let $\mathcal{B}'$ be a cover of $\overline{A}$ with $\mu(\mathcal{B}') \le \mu(\overline{\mathcal{A}}) + \delta$. Use Lemma 4A.2 to show that $\mu(\mathcal{B} \cap \mathcal{B}') = \mu(B) + \mu(\mathcal{B}') - T$.

(c) Show that $\mu^{\rm o}(B \cap \overline{A}) \le \mu(B \cap \mathcal{B}') \le \mu(\mathcal{B}) - \mu(A) + \delta$.

(d) Show that $\mu^{\rm o}(B \cap \overline{A}) = \mu(B) - \mu(A)$.

4.10. (Intersection of covers) Let $\mathcal{A}$ be an arbitrary set in $[-T/2, T/2]$.

(a) Show that $\mathcal{A}$ has a sequence of covers, $\mathcal{B}_1, \mathcal{B}_2, \dots$ such that $\mu^{\rm o}(\mathcal{A}) = \mu(\mathcal{D})$ where $\mathcal{D} = \bigcap_n \mathcal{B}_n$.

(b) Show that $\mathcal{A} \subseteq \mathcal{D}$.

(c) Show that if $\mathcal{A}$ is measurable, then $\mu(\mathcal{D} \cap \overline{\mathcal{A}}) = 0$. Note that you have shown that an arbitrary measurable set can be represented as a countable intersection of countable unions of intervals, less a set of zero measure. Argue by example that if $\mathcal{A}$ is not measurable, then $\mu^{\mathrm{o}}(\mathcal{D} \cap \overline{\mathcal{A}})$ need not be 0.

4.11. (Measurable functions) (a) For $\{u(t) : [-T/2, T/2] \to R\}$, show that if $\{t : u(t) < \beta\}$ is measurable, then $\{t : u(t) \geq \beta\}$ is measurable.

(b) Show that if $\{t : u(t) < \beta\}$ and $\{t : u(t) < \alpha\}$ are measurable, $\alpha < \beta$, then $\{t : \alpha \leq u(t) < \beta\}$ is measurable.

(c) Show that if $\{t : u(t) < \beta\}$ is measurable for all β, then $\{t : u(t) \leq \beta\}$ is also measurable. Hint: Express $\{t : u(t) \leq \beta\}$ as a countable intersection of measurable sets.

(d) Show that if $\{t : u(t) \leq \beta\}$ is measurable for all β, then $\{t : u(t) < \beta\}$ is also measurable, *i.e.*, the definition of measurable function can use either strict or nonstrict inequality.

4.12. (Measurable functions) Assume throughout that $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$ is measurable.

(a) Show that $-u(t)$ and $|u(t)|$ are measurable.

(b) Assume that $\{g(x) : \mathbb{R} \to \mathbb{R}\}$ is an increasing function (*i.e.*, $x_1 < x_2 \implies g(x_1) < g(x_2)$). Prove that $v(t) = g(u(t))$ is measurable Hint: This is a one liner. If the abstraction confuses you, first show that $\exp(u(t))$ is measurable and then prove the more general result.

(c) Show that $\exp[u(t)], u^2(t)$, and $\ln |u(t)|$ are all measurable.

4.13. (Measurable functions) (a) Show that if$\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$ and $\{v(t) : [-T/2, T/2] \to \mathbb{R}\}$ are measurable, then $u(t) + v(t)$ is also measurable. Hint: Use a discrete approximation to the sum and then go to the limit.

(b) Show that $u(t)v(t)$ is also measurable.

4.14. (Measurable sets) Suppose $\mathcal{A}$ is a subset of $[-T/2, T/2]$ and is measurable over $[-T/2, T/2]$. Show that $\mathcal{A}$ is also measurable, with the same measure, over $[-T'/2, T'/2]$ for any T' satisfying $T' > T$. Hint: Let $\mu'(\mathcal{A})$ be the outer measure of $\mathcal{A}$ over $[-T'/2, T'/2]$ and show that $\mu'(\mathcal{A}) = \mu^{\mathrm{o}}(\mathcal{A})$ where μ^{o} is the outer measure over $[-T/2, T/2]$. Then let $\overline{\mathcal{A}}'$ be the complement of $\mathcal{A}$ over $[-T'/2, T'/2]$ and show that $\mu'(\overline{\mathcal{A}}') = \mu^{\mathrm{o}}(\overline{\mathcal{A}}) + T' - T$.

4.15. (Measurable limits) (a) Assume that $\{u_n(t) : [-T/2, T/2] \to \mathbb{R}\}$ is measurable for each $n \geq 1$. Show that $\liminf_n u_n(t)$ is measurable ($\liminf_n u_n(t)$ means $\lim_m v_m(t)$ where $v_m(t) = \inf_{n=m}^{\infty} u_n(t)$ and infinite values are allowed).

(b) Show that $\lim_n u_n(t)$ exists for a given t if and only if $\liminf_n u_n(t) = \limsup_n u_n(t)$.

(c) Show that the set of t for which $\lim_n u_n(t)$ exists is measurable. Show that a function $u(t)$ that is $\lim_n u_n(t)$ when the limit exists and is 0 otherwise is measurable.

4.16. (Lebesgue integration) For each integer $n \geq 1$, define $u_n(t) = 2^n \,\mathrm{rect}(2^n t - 1)$. Sketch the first few of these waveforms. Show that $\lim_{n\to\infty} u_n(t) = 0$ for all t. Show that $\int \lim_n u_n(t)\, dt \neq \lim_n \int u_n(t)\, dt$.

4.17. ($\mathcal{L}_1$ integrals)) (a) Assume that $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$ is $\mathcal{L}_1$. Show that

$$\left|\int u(t)\, dt\right| = \left|\int u^+(t)\, dt - \int u^-(t)\, dt\right| \leq \int |u(t)|\, dt.$$

(b) Assume that $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is $\mathcal{L}_1$. Show that

$$\left| \int u(t)\, dt \right| \leq \int |u(t)|\, dt.$$

Hint: Choose α such that $\alpha \int u(t)\, dt$ is real and nonnegative and $|\alpha| = 1$. Use part (a) on $\alpha u(t)$.

4.18. ($\mathcal{L}_2$ equivalence) Assume that $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ and $\{v(t) : [-T/2, T/2] \to \mathbb{C}\}$ are $\mathcal{L}_2$ functions.

(a) Show that if $u(t)$ and $v(t)$ are equal a.e., then they are $\mathcal{L}_2$ equivalent.

(b) Show that if $u(t)$ and $v(t)$ are $\mathcal{L}_2$ equivalent, then for any $\varepsilon > 0$, the set $\{t : |u(t) - v(t)|^2 \geq \varepsilon\}$ has zero measure.

(c) Using (b), show that $\mu\{t : |u(t) - v(t)| > 0\} = 0$, *i.e.*, that $u(t) = v(t)$ a.e.

4.19. (Orthogonal expansions) Assume that $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is $\mathcal{L}_2$. Let $\{\theta_k(t); 1 \leq k < \infty\}$ be a set of orthogonal waveforms and assume that $u(t)$ has the orthogonal expansion

$$u(t) = \sum_{k=1}^{\infty} u_k \theta_k(t).$$

Assume the set of orthogonal waveforms satisfy

$$\int_{-\infty}^{\infty} \theta_k(t)\theta_j^*(t)\, dt = \begin{cases} 0 & \text{for} \quad k \neq j \\ A_j & \text{for} \quad k = j, \end{cases}$$

where $\{A_j\}$ is an arbitrary set of positive numbers. Do not concern yourself with convergence issues in this exercise.

(a) Show that each u_k can be expressed in terms of $\int_{-\infty}^{\infty} u(t)\theta_k^*(t)\, dt$ and A_k.

(b) Find the energy $\int_{-\infty}^{\infty} |u(t)|^2 dt$ in terms of $\{u_k\}$, and $\{A_k\}$.

(c) Suppose that $v(t) = \sum_k v_k \theta_k(t)$ where $v(t)$ also has finite energy. Express $\int_{-\infty}^{\infty} u(t)v^*(t)\, dt$ as a function of $\{u_k, v_k, A_k; k \in \mathbb{Z}\}$.

4.20. (Fourier series) (a) Verify that (4.22) and (4.23) follow from (4.20) and (4.18) using the transformation $u(t) = v(t + \Delta)$.

(b) Consider the Fourier series in periodic form, $w(t) = \sum_k \hat{w}_k e^{2\pi ikt/T}$ where $\hat{w}_k = (1/T) \int_{-T/2}^{T/2} w(t)e^{-2\pi ikt/T}\, dt$. Show that for any real Δ, $(1/T) \int_{-T/2+\Delta}^{T/2+\Delta} w(t)e^{-2\pi ikt/T}\, dt$ is also equal to $\hat{w}_k$, providing an alternate derivation of (4.22) and (4.23).

4.21. Equation (4.27) claims that

$$\lim_{n\to\infty, \ell\to\infty} \int \left| u(t) - \sum_{m=-n}^{n} \sum_{k=-\ell}^{\ell} \hat{u}_{k,m}\theta_{k,m}(t) \right|^2 dt = 0$$

(a) Show that the integral above is non-increasing in both ℓ and n.

(b) Show that the limit is independent of how n and ℓ approach ∞. Hint: See Exercise 4.4.

(c) More generally, show that the limit is the same if the pair (k, m), $k \in \mathbb{Z}, m \in \mathbb{Z}$ is ordered in an arbitrary way and the limit above is replaced by a limit on the partial sums according to that ordering.

4.22. (Truncated sinusoids) (a) Verify (4.24) for $\mathcal{L}_2$ waveforms, *i.e.*, show that

$$\lim_{n\to\infty} \int \left|u(t) - \sum_{m=-n}^{n} u_m(t)\right|^2 dt = 0.$$

(b) Break the integral in (4.28) into separate integrals for $|t| > (n+\frac{1}{2})T$ and $|t| \le (n+\frac{1}{2})T$. Show that the first integral goes to 0 with increasing n.

(c) For given n, show that the second integral above goes to 0 with increasing ℓ.

4.23. (Convolution) The left side of (4.40) is a function of t. Express the Fourier transform of this as a double integral over t and τ. For each t, make the substitution $r = t - \tau$ and integrate over r. Then integrate over τ to get the right side of (4.40). Do not concern yourself with convergence issues here.

4.24. (Continuity of $\mathcal{L}_1$ transform) Assume that $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is $\mathcal{L}_1$ and let $\hat{u}(f)$ be its Fourier transform. Let ε be any given positive number.

(a) Show that for sufficiently large T, $\int_{|t|>T} |u(t)e^{-2\pi ift} - u(t)e^{-2\pi i(f-\delta)t}|\, dt < \varepsilon/2$ for all f and all $\delta > 0$.

(b) For the ε and T selected above, show that $\int_{|t|\le T} |u(t)e^{-2\pi ift} - u(t)e^{-2\pi i(f-\delta)t}|\, dt < \varepsilon/2$ for all f and sufficiently small $\delta > 0$. This shows that $\hat{u}(f)$ is continuous.

4.25. (Plancherel) The purpose of this exercise is to get some understanding of the Plancherel theorem. Assume that $u(t)$ is $\mathcal{L}_2$ and has a Fourier transform $\hat{u}(f)$.

(a) Show that $\hat{u}(f) - \hat{u}_A(f)$ is the Fourier transform of the function $x_A(t)$ that is 0 from $-A$ to A and equal to $u(t)$ elsewhere.

(b) Argue that since $\int_{-\infty}^{\infty} |u(t)|^2 dt$ is finite, the integral $\int_{-\infty}^{\infty} |x_A(t)|^2\, dt$ must go to 0 as $A \to \infty$. Use whatever level of mathematical care and common sense that you feel comfortable with.

(c) Using the energy equation (4.45), argue that

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} |\hat{u}(f) - \hat{u}_A(f)|^2\, dt = 0.$$

Note: This is only the easy part of the Plancherel theorem. The difficult part is to show the existence of $\hat{u}(f)$. The limit as $A \to \infty$ of the integral $\int_{-A}^{A} u(t)e^{-2\pi ift}\, dt$ need not exist for all f, and the point of the Plancherel theorem is to forget about this limit for individual f and focus instead on the energy in the difference between the hypothesized $\hat{u}(f)$ and the approximations.

4.26. ($\mathcal{L}_2$ functions) Assume that $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ and $\{v(t) : \mathbb{R} \to \mathbb{C}\}$ are $\mathcal{L}_2$ and that a and b are complex numbers. Show that $au(t) + bv(t)$ is $\mathcal{L}_2$. For $T > 0$, show that $u(t - T)$ and $u(\frac{t}{T})$ are $\mathcal{L}_2$ functions.

4.27. (Relation of Fourier series to Fourier integral) Assume that $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is $\mathcal{L}_2$. Without being very careful about the mathematics, the Fourier series expansion of $\{u(t)\}$ is given by

$$u(t) = \lim_{\ell\to\infty} u^{(\ell)}(t) \quad \text{where} \quad u^{(\ell)}(t) = \sum_{k=-\ell}^{\ell} \hat{u}_k e^{2\pi ikt/T}\, \mathrm{rect}(\frac{t}{T})$$

$$\hat{u}_k = \frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-2\pi ikt/T}\, dt.$$

(a) Does the above limit hold for all $t \in [-T/2, T/2]$? If not, what can you say about the type of convergence?

(b) Does the Fourier transform $\hat{u}(f) = \int_{-T/2}^{T/2} u(t)e^{-2\pi i f t}\, dt$ exist for all f? Explain.

(c) The Fourier transform of the finite sum $u^{(\ell)}(t)$ is $\hat{u}^{(\ell)}(f) = \sum_{k=-\ell}^{\ell} \hat{u}_k T \text{sinc}(fT - k)$. In the limit $\ell \to \infty$, $\hat{u}(f) = \lim_{\ell\to\infty} \hat{u}^{(\ell)}(f)$, so

$$\hat{u}(f) = \lim_{\ell\to\infty} \sum_{k=-\ell}^{\ell} \hat{u}_k T \,\text{sinc}(fT - k).$$

Give a brief explanation why this equation must hold with equality for all $f \in \mathbb{R}$. Also show that $\{\hat{u}(f) : f \in \mathbb{R}\}$ is completely specified by its values, $\{\hat{u}(k/T) : k \in \mathbb{Z}\}$ at multiples of $1/T$.

4.28. (sampling) One often approximates the value of an integral by a discrete sum; *i.e.*,

$$\int_{-\infty}^{\infty} g(t)\, dt \approx \delta \sum_k g(k\delta).$$

(a) Show that if $u(t)$ is a real finite-energy function, low-pass limited to W Hz, then the above approximation is exact for $g(t) = u^2(t)$ if $\delta \le 1/(2\mathsf{W})$; *i.e.*, show that

$$\int_{-\infty}^{\infty} u^2(t)\, dt = \delta \sum_k u^2(k\delta).$$

(b) Show that if $g(t)$ is a real finite-energy function, low-pass limited to W Hz, then for $\delta \le 1/(2\mathsf{W})$,

$$\int_{-\infty}^{\infty} g(t)\, dt = \delta \sum_k g(k\delta).$$

(c) Show that if $\delta > 1/2\mathsf{W}$, then there exists no such relation in general.

4.29. (degrees of freedom) This exercise explores how much of the energy of a baseband-limited function $\{u(t) : [-1/2, 1/2] \to \mathbb{R}\}$ can reside outside the region where the sampling coefficients are nonzero. Let $T = 1/(2\mathsf{W}) = 1$ and let n be a given positive even integer. Let $u_k = (-1)^k$ for $-n \le k \le n$ and $u_k = 0$ for $|k| > n$. Show that $|u(n+\frac{1}{2})|$ increases without bound as the end point n is increased. Show that $|u(n+m+\frac{1}{2})| > |u(n-m-\frac{1}{2})|$ for all integer m, $0 \le m < n$. In other words, shifting the sample points by 1/2 leads to most of the sample point energy being outside the interval $[-n, n]$.

4.30. (sampling theorem for $[\Delta - \mathsf{W}, \Delta + \mathsf{W})]$) (a) Verify the Fourier transform pair in (4.70). Hint: Use the scaling and shifting rules on $\text{rect}(f) \leftrightarrow \text{sinc}(t)$.

(b) Show that the functions making up that expansion are orthogonal. Hint: Show that the corresponding Fourier transforms are orthogonal.

(c) Show that the functions in (4.74) are orthogonal.

4.31. (Amplitude limited functions) Sometimes it is important to generate baseband waveforms with bounded amplitude. This problem explores pulse shapes that can accomplish this

(a) Find the Fourier transform of $g(t) = \text{sinc}^2(\mathsf{W}t)$. Show that $g(t)$ is bandlimited to $f \le \mathsf{W}$ and sketch both $g(t)$ and $\hat{g}(f)$. (Hint: Recall that multiplication in the time domain corresponds to convolution in the frequency domain.)

(b) Let $u(t)$ be a continuous real $\mathcal{L}_2$ function baseband limited to $f \le \mathsf{W}$ (*i.e.*, a function such that $u(t) = \sum_k u(kT)\text{sinc}\,(t/T - k)$ where $T = 1/2\mathsf{W}$). Let $v(t) = u(t) * g(t)$. Express $v(t)$ in terms of the samples $\{u(kT); k \in \mathbb{Z}\}$ of $u(t)$ and the shifts $\{g(t - kT); k \in \mathbb{Z}\}$ of $g(t)$. Hint: Use your sketches in part (a) to evaluate $g(t) * \text{sinc}(t/T)$.

(c) Show that if the T-spaced samples of $u(t)$ are non-negative, then $v(t) \ge 0$ for all t.

(d) Explain why $\sum_k \text{sinc}(t/T - k) = 1$ for all t.

(e) Using (d), show that $\sum_k g(t - kT) = c$ for all t and find the constant c. Hint: Use the hint in (b) again.

(f) Now assume that $u(t)$, as defined in part (b), also satisfies $u(kT) \le 1$ for all $k \in \mathbb{Z}$. Show that $v(t) \le 2$ for all t.

(g) Allow $u(t)$ to be complex now, with $|u(kT)| \le 1$. Show that $|v(t)| \le 2$ for all t.

4.32. (Orthogonal sets) The function $\text{rect}(t/T)$ has the very special property that it, plus its time and frequency shifts, by kT and j/T respectively, form an orthogonal set. The function $\text{sinc}(t/T)$ has this same property. We explore other functions that are generalizations of $\text{rect}(t/T)$ and which, as you will show in parts (a) to (d), have this same interesting property. For simplicity, choose $T = 1$.

These functions take only the values 0 and 1 and are allowed to be non-zero only over [-1, 1] rather than $[-1/2, 1/2]$ as with $\text{rect}(t)$. Explicitly, the functions considered here satisfy the following constraints:

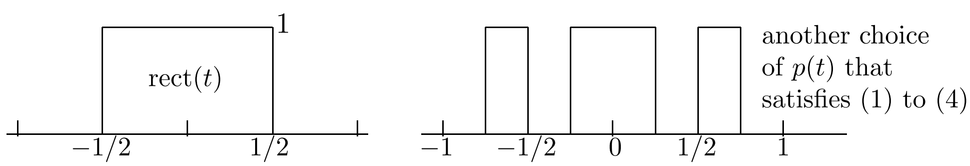
$$p(t) = p^2(t) \qquad \text{for all } t \quad \text{(0/1 property)} \tag{4.108}$$

$$p(t) = 0 \qquad \text{for } |t| > 1 \tag{4.109}$$

$$p(t) = p(-t) \qquad \text{for all } t \quad \text{(symmetry)} \tag{4.110}$$

$$p(t) = 1 - p(t-1) \qquad \text{for } 0 \le t < 1/2. \tag{4.111}$$

Note: Because of property (4.110), condition (4.111) also holds for $1/2 < t \le 1$. Note also that $p(t)$ at the single points $t = \pm 1/2$ does not effect any orthogonality properties, so you are free to ignore these points in your arguments.



(a) Show that $p(t)$ is orthogonal to $p(t-1)$. Hint: evaluate $p(t)p(t-1)$ for each $t \in [0, 1]$ other than $t = 1/2$.

(b) Show that $p(t)$ is orthogonal to $p(t-k)$ for all integer $k \ne 0$.

(c) Show that $p(t)$ is orthogonal to $p(t-k)e^{i2\pi mt}$ for integer $m \ne 0$ and $k \ne 0$.

(d) Show that $p(t)$ is orthogonal to $p(t)e^{2\pi imt}$ for integer $m \ne 0$. Hint: Evaluate $p(t)e^{-2\pi imt} + p(t-1)e^{-2\pi im(t-1)}$.

(e) Let $h(t) = \hat{p}(t)$ where $\hat{p}(f)$ is the Fourier transform of $p(t)$. If $p(t)$ satisfies properties (1) to (4), does it follow that $h(t)$ has the property that it is orthogonal to $h(t-k)e^{2\pi imt}$ whenever either the integer k or m is non-zero?

Note: Almost no calculation is required in this problem.

4.33. (limits) Construct an example of a sequence of $\mathcal{L}_2$ functions $v^{(m)}(t), m \in \mathbb{Z}, m > 0$ such that $\lim_{m\to\infty} v^{(m)}(t) = 0$ for all t but for which $\underset{m\to\infty}{\text{l.i.m.}}\ v^{(m)}(t)$ does not exist. In other words show that pointwise convergence does not imply $\mathcal{L}_2$ convergence. Hint: Consider time shifts.

4.34. (aliasing) Find an example where $\hat{u}(f)$ is 0 for $|f| > 3\mathsf{W}$ and nonzero for $\mathsf{W} < |f| < 3\mathsf{W}$ but where, with $T = 1/(2\mathsf{W})$, $s(kT) = v_0(kT)$ (as defined in (4.77)) for all $k \in \mathbb{Z}$). Hint: Note that it is equivalent to achieve equality between $\hat{s}(f)$ and $\hat{u}(f)$ for $|f| \le \mathsf{W}$. Look at Figure 4.10.

4.35. (aliasing) The following exercise is designed to illustrate the sampling of an approximately baseband waveform. To avoid messy computation, we look at a waveform baseband-limited to 3/2 which is sampled at rate 1 (*i.e.*, sampled at only 1/3 the rate that it should be sampled at). In particular, let $u(t) = \text{sinc}(3t)$.

(a) Sketch $\hat{u}(f)$. Sketch the function $\hat{v}_m(f) = \text{rect}(f-m)$ for each integer m such that $v_m(f) \neq 0$. Note that $\hat{u}(f) = \sum_m \hat{v}_m(f)$.

(b) Sketch the inverse transforms $v_m(t)$ (real and imaginary part if complex).

(c) Verify directly from the equations that $u(t) = \sum v_m(t)$. Hint: this is easiest if you express the sine part of the sinc function as a sum of complex exponentials.

(d) Verify the sinc-weighted sinusoid expansion, (4.73). (There are only 3 nonzero terms in the expansion.)

(e) For the approximation $s(t) = u(0)\text{sinc}(t)$, find the energy in the difference between $u(t)$ and $s(t)$ and interpret the terms.

4.36. (aliasing) Let $u(t)$ be the inverse Fourier transform of a function $\hat{u}(f)$ which is both $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $v_m(t) = \int \hat{u}(f)\text{rect}(fT-m)e^{2\pi ift}\,df$ and let $v^{(n)}(t) = \sum_{-n}^{n} v_m(t)$.

(a) Show that $|u(t) - v^{(n)}(t)| \le \int_{|f|\ge(2n+1)/T} |\hat{u}(f)|\,df$ and thus that $u(t) = \lim_{n\to\infty} v^{(n)}(t)$ for all t.

(b) Show that the sinc-weighted sinusoid expansion of (4.76) then converges pointwise for all t. Hint: for any t and any $\varepsilon > 0$, choose n so that $|u(t) - v^n(t)| \le \varepsilon/2$. Then for each m, $|m| \le n$, expand $v_m(t)$ in a sampling expansion using enough terms to keep the error less than $\frac{\varepsilon}{4n+2}$.

4.37. (aliasing) (a) Show that $\hat{s}(f)$ in (4.83) is $\mathcal{L}_1$ if $\hat{u}(f)$ is.

(b) Let $\hat{u}(f) = \sum_{k\neq 0} \text{rect}[k^2(f-k)]$. Show that $\hat{u}(f)$ is $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $T = 1$ for $\hat{s}(f)$ and show that $\hat{s}(f)$ is not $\mathcal{L}_2$. Hint: Sketch $\hat{u}(f)$ and $\hat{s}(f)$.

(c) Show that $\hat{u}(f)$ does not satisfy $\lim_{|f|\to\infty} \hat{u}(f)|f|^{1+\varepsilon} = 0$.

4.38. (aliasing) Let $u(t) = \sum_{k\neq 0} \text{rect}[k^2(t-k)]$ and show that $u(t)$ is $\mathcal{L}_2$. Find $s(t) = \sum_k u(k)\text{sinc}(t-k)$ and show that it is neither $\mathcal{L}_1$ nor $\mathcal{L}_2$. Find $\sum_k u^2(k)$ and explain why the sampling theorem energy equation (4.66) does not apply here.