Problem 3.1 (Invariance of coding gain)

(a) Show that in the power-limited regime the nominal coding gain $\gamma_c(\mathcal{A})$ of (5.9), the UBE (5.10) of $P_b(E)$, and the effective coding gain $\gamma_{\text{eff}}(\mathcal{A})$ are invariant to scaling, orthogonal transformations and Cartesian products.

(b) Show that in the bandwidth-limited regime the nominal coding gain $\gamma_c(\mathcal{A})$ of (5.14), the UBE (5.15) of $P_b(E)$, and the effective coding gain $\gamma_{\text{eff}}(\mathcal{A})$ are invariant to scaling, orthogonal transformations and Cartesian products.

Problem 3.2 (Orthogonal signal sets)

An orthogonal signal set is a set $\mathcal{A} = \{\mathbf{a}_j, 1 \leq j \leq M\}$ of $M$ orthogonal vectors in $\mathbb{R}^M$ with equal energy $E(\mathcal{A})$; i.e., $\langle \mathbf{a}_j, \mathbf{a}_{j'} \rangle = E(\mathcal{A})\delta_{jj'}$ (Kronecker delta).

(a) Compute the nominal spectral efficiency $\rho$ of $\mathcal{A}$ in bits per two dimensions. Compute the average energy $E_b$ per information bit.

(b) Compute the minimum squared distance $d^2_{\text{min}}(\mathcal{A})$. Show that every signal has $K_{\text{min}}(\mathcal{A}) = M - 1$ nearest neighbors.

(c) Let the noise variance be $\sigma^2 = N_0/2$ per dimension. Show that the probability of error of an optimum detector is bounded by the UBE

$$\Pr(E) \leq (M - 1)Q\sqrt{\frac{E(\mathcal{A})}{N_0}}.$$  

(d) Let $M \to \infty$ with $E_b$ held constant. Using an asymptotically accurate upper bound for the $Q(\cdot)$ function (see Appendix), show that $\Pr(E) \to 0$ provided that $E_b/N_0 > 2\ln 2$ (1.42 dB). How close is this to the ultimate Shannon limit on $E_b/N_0$? What is the nominal spectral efficiency $\rho$ in the limit?

Problem 3.3 (Simplex signal sets)

Let $\mathcal{A}$ be an orthogonal signal set as above.

(a) Denote the mean of $\mathcal{A}$ by $\mathbf{m}(\mathcal{A})$. Show that $\mathbf{m}(\mathcal{A}) \neq \mathbf{0}$, and compute $||\mathbf{m}(\mathcal{A})||^2$.

The zero-mean set $\mathcal{A}' = \mathcal{A} - \mathbf{m}(\mathcal{A})$ (as in Exercise 2) is called a simplex signal set. It is universally believed to be the optimum set of $M$ signals in AWGN in the absence of bandwidth constraints, except at ridiculously low SNRs.

(b) For $M = 2, 3, 4$, sketch $\mathcal{A}$ and $\mathcal{A}'$.

(c) Show that all signals in $\mathcal{A}'$ have the same energy $E(\mathcal{A}')$. Compute $E(\mathcal{A}')$. Compute the inner products $\langle \mathbf{a}_j, \mathbf{a}_{j'} \rangle$ for all $\mathbf{a}_j, \mathbf{a}_{j'} \in \mathcal{A}'$. 

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Problem 3.4 (Biorthogonal signal sets)
The set $\mathcal{A}'' = \pm \mathcal{A}$ of size $2M$ consisting of the $M$ signals in an orthogonal signal set $\mathcal{A}$ with symbol energy $E(\mathcal{A})$ and their negatives is called a biorthogonal signal set.

(a) Show that the mean of $\mathcal{A}''$ is $\mathbf{m}(\mathcal{A}'') = \mathbf{0}$, and that the average energy is $E(\mathcal{A})$.
(b) How much greater is the nominal spectral efficiency $\rho$ of $\mathcal{A}''$ than that of $\mathcal{A}$?
(c) Show that the probability of error of $\mathcal{A}''$ is approximately the same as that of an orthogonal signal set with the same size and average energy, for $M$ large.
(d) Let the number of signals be a power of 2: $2M = 2^k$. Show that the nominal spectral efficiency is $\rho(\mathcal{A}'') = 4k2^{-k} b/2D$, and that the nominal coding gain is $\gamma_c(\mathcal{A}'') = k/2$. Show that the number of nearest neighbors is $K_{\text{min}}(\mathcal{A}'') = 2^k - 2$.

Problem 3.5 (small nonbinary constellations)

(a) For $M = 4$, the (2 × 2)-QAM signal set is known to be optimal in $N = 2$ dimensions. Show however that there exists at least one other inequivalent two-dimensional signal set $\mathcal{A}'$ with the same coding gain. Which signal set has the lower “error coefficient” $K_{\text{min}}(\mathcal{A})$? [Hint: consider the signal set $\mathcal{A}'' = \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}$.] Sketch $\mathcal{A}''$. What is the geometric name of the polytope whose vertex set is $\mathcal{A}''$?

(b) Show that the coding gain of (a) can be improved in $N = 3$ dimensions. [Hint: consider the signal set $\mathcal{A}''' = \{(1, 1, 1), (1, -1, 1), (1, -1, -1), (1, -1, 1)\}$.] Sketch $\mathcal{A}'''$. Give an accurate plot of the UBE of the Pr$(E)$ for the signal set $\mathcal{A}'''$ of (b). How much is the effective coding gain, by our rule of thumb and by this plot?

(d) For $M = 8$ and $N = 2$, propose at least two good signal sets, and determine which one is better. [Open research problem: Find the optimal such signal set, and prove that it is optimal.]

Problem 3.6 (Even-weight codes have better coding gain)

Let $\mathcal{C}$ be an $(n, k, d)$ binary linear code with $d$ odd. Show that if we append an overall parity check $p = \sum_i x_i$ to each codeword $\mathbf{x}$, then we obtain an $(n + 1, k, d + 1)$ binary linear code $\mathcal{C}'$ with $d$ even. Show that the nominal coding gain $\gamma_c(\mathcal{C}')$ is always greater than $\gamma_c(\mathcal{C})$ if $k > 1$. Conclude that we can focus primarily on linear codes with $d$ even.