Problem 3.1
Here we shall extend the results of Problem 2.2 to include classically-random polarizations. Suppose we have a $+z$-propagating, frequency-$\omega$ photon whose polarization vector (in Problem 2.1 notation) is,

$$\mathbf{i} \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

where $\alpha_x$ and $\alpha_y$ are a pair of complex-valued classical random variables that satisfy

$$|\alpha_x|^2 + |\alpha_y|^2 = 1,$$

with probability one. (Two joint complex-valued random variables, $\alpha_x$ and $\alpha_y$, are really four joint real-valued random variables, viz., the real and imaginary parts of $\alpha_x$ and $\alpha_y$.)

The Poincaré sphere representation for the average behavior of this random polarization vector is,

$$\mathbf{r} \equiv \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2\text{Re}[\langle \alpha_x^* \alpha_y \rangle] \\ 2\text{Im}[\langle \alpha_x^* \alpha_y \rangle] \\ \langle |\alpha_x|^2 \rangle - \langle |\alpha_y|^2 \rangle \end{bmatrix},$$

where—in keeping with the quantum notation for averages—$\langle \cdot \rangle$ denotes ensemble average.

(a) Use the Schwarz inequality to prove that $\mathbf{r}^T \mathbf{r} \equiv r_1^2 + r_2^2 + r_3^2 \leq 1$, i.e., the $\mathbf{r}$ vector lies on or inside the unit sphere.

(b) Let $\mathbf{i}_a$ and $\mathbf{i}_b$ be a pair of deterministic, orthogonal, complex-valued unit vectors, viz.,

$$\mathbf{i}_k^\dagger \mathbf{i}_l = \delta_{kl} \equiv \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases},$$

where $k$ and $l$ can each be either $a$ or $b$. By means of wave plates, a polarizing beam splitter, and a pair of ideal photon counters, it is possible to measure whether the photon is polarized along $\mathbf{i}_a$ or along $\mathbf{i}_b$, by which we mean whether
the $i_a$-polarization or the $i_b$-polarization detector is the one that registers a photon detection. The statistics of this measurement satisfy,

\[
\text{Pr(}\text{polarized along } i_a\text{)} = \frac{1 + r_a^T r}{2}, \quad (1)
\]

\[
\text{Pr(}\text{polarized along } i_b\text{)} = \frac{1 + r_b^T r}{2}, \quad (2)
\]

where $r_a$ and $r_b$ are the Poincaré sphere representations of $i_a$ and $i_b$, respectively.

Use the orthogonality of $i_a$ and $i_b$ to show that $r_a = -r_b$, so that Eqs. (1) and (2) constitute a proper probability distribution.

(c) Suppose that the photon’s random polarization leads to $r = 0$, i.e., $r_1 = r_2 = r_3 = 0$. Show that

\[
\text{Pr(}\text{polarized along } i_a\text{)} = \text{Pr(}\text{polarized along } i_b\text{)} = \frac{1}{2},
\]

for all pairs of deterministic, orthogonal complex-valued unit vectors $\{i_a, i_b\}$, and thus that $r = 0$ represents a state of completely random polarization. Contrast the preceding measurement statistics with what will be obtained when

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \quad r_a = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \quad r_b = \begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix},
\]

and when

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \quad r_a = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \quad r_b = \begin{bmatrix}
0 \\
-1 \\
0
\end{bmatrix},
\]

are the Poincaré sphere representations of the photon and the pair of orthogonal polarizations being measured.

**Problem 3.2**

Here we introduce the notion of a density operator, i.e., a way to account for classical randomness limiting our knowledge of a quantum system’s state. Consider a quantum mechanical system whose state is not known. Instead, there is a classical probability distribution for this state. In particular, suppose that there are $M$ distinct unit-length kets, $\{ |\psi_m\rangle : 1 \leq m \leq M \}$, and that the system is known to be in one of these states. Moreover the probability that it is in state $|\psi_m\rangle$ is $p_m$, for $1 \leq m \leq M$, where $\{ p_m : 1 \leq m \leq M \}$ is a classical probability distribution: $p_m \geq 0$ and $\sum_{m=1}^{M} p_m = 1$. 

2
(a) Suppose that we measure the observable \( \hat{O} \), where \( \hat{O} \) has distinct eigenvalues, \( \{ o_n : 1 \leq n < \infty \} \), and a complete orthonormal set of associated eigenkets, \( \{ |o_n\rangle : 1 \leq n < \infty \} \). GIVEN that the state of the system is \( |\psi_m\rangle \), we know that the \( \hat{O} \) measurement will yield outcome \( o_n \) with conditional probability \( \Pr(o_n | |\psi_m\rangle) \equiv |\langle o_n|\psi_m\rangle|^2 \), for \( 1 \leq n < \infty \) and \( 1 \leq m \leq M \). Use this conditional probability distribution to obtain the unconditional probability, \( \Pr(o_n) \), of getting the outcome \( o_n \) when we make the \( \hat{O} \) measurement.

(b) Define a density operator for the system by,
\[
\hat{\rho} \equiv \sum_{m=1}^{M} p_m |\psi_m\rangle\langle\psi_m|.
\]
Show that \( \hat{\rho} \) is an Hermitian operator, and verify that your answer to (a) can be reduced to
\[
\Pr(o_n) = \langle o_n | \hat{\rho} | o_n \rangle, \quad \text{for} \quad 1 \leq n < \infty.
\]

(c) Show that the expected value of the \( \hat{O} \) measurement, i.e.,
\[
\langle \hat{O} \rangle \equiv \sum_{n=1}^{\infty} o_n \Pr(o_n),
\]
satisfies
\[
\langle \hat{O} \rangle = \text{tr}(\hat{\rho} \hat{O}),
\]
where \( \text{tr}(\hat{A}) \) for any linear Hilbert-space operator, \( \hat{A} \), is the trace of that operator, defined as follows. Let \( \{ |k\rangle : 1 \leq k < \infty \} \) be an arbitrary complete set of orthonormal kets on the quantum system’s state space, so that
\[
\hat{I} = \sum_{k=1}^{\infty} |k\rangle\langle k|.
\]
Then
\[
\text{tr}(\hat{A}) \equiv \sum_{k=1}^{\infty} \langle k|\hat{A}|k\rangle,
\]
i.e., it is the sum of the operator’s diagonal matrix-elements in the \( \{ |k\rangle \} \) representation. **Comment:** The trace operation is invariant to the choice of the CON basis used for its calculation. Hence a propitious choice of the basis can be a great aid in simplifying the calculation of averages involving a density operator.

**Problem 3.3**
Here we will explore the difference between a pure state and a mixed state, i.e., the difference between knowing that a quantum system is in a definite state \( |\psi\rangle \) as
opposed to having a classically-random distribution over a set of such states, namely a density operator $\hat{\rho}$. Because the density operator is Hermitian, it has eigenvalues and eigenkets. Let us assume that these form a countable set, viz., $\hat{\rho}$ has eigenvalues, $\{ \rho_n : 1 \leq n < \infty \}$, and associated eigenkets $\{ |\rho_n\rangle : 1 \leq n < \infty \}$, that satisfy

$$\hat{\rho} |\rho_n\rangle = \rho_n |\rho_n\rangle, \quad \text{for} \ 1 \leq n < \infty.$$ 

Without loss of generality, we can assume that these eigenkets form a complete orthonormal set, i.e.,

$$\langle \rho_m | \rho_n \rangle = \delta_{nm},$$

$$\hat{I} = \sum_{n=1}^{\infty} |\rho_n\rangle \langle \rho_n|.$$

(a) Show that the eigenvalues $\{\rho_n\}$ satisfy

$$0 \leq \rho_n \leq 1, \quad \text{for} \ 1 \leq n < \infty,$$

and

$$\sum_{n=1}^{\infty} \rho_n = 1.$$

(b) Show that $\text{tr}(\hat{\rho}) = 1$ for any density operator

(c) Suppose that the quantum system is in a pure state, i.e., it is known to be in the state $|\psi\rangle$. Show that this situation can be represented in density-operator form by setting $\rho_1 = 1$ and $|\rho_1\rangle = |\psi\rangle$, viz., a pure state has a density operator with only one eigenket whose associated eigenvalue is non-zero. Show that $\text{tr}(\hat{\rho}^2) = 1$ for any pure-state density operator.

(d) When the density operator has two or more eigenkets with non-zero eigenvalues we say that the state is mixed, i.e., there are at least two different pure states that can occur with non-zero probabilities. Show that $\text{tr}(\hat{\rho}^2) < 1$ for any mixed-state density operator.

**Problem 3.4**

In this problem we shall explore the density operator for single-photon polarization. Suppose that we are interested in the polarization state of a frequency-ω, +z-propagating, single photon. We know that a pure state of such a photon can be written as the 2-D complex-valued ket vector,

$$|i\rangle \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}.$$
in the $x$-$y$ (horizontal-vertical) basis, with $|\alpha_x|^2 + |\alpha_y|^2 = 1$. If we measure the polarization state of this photon using the basis,

$$|i'\rangle \equiv \begin{bmatrix} \alpha'_x \\ \alpha'_y \end{bmatrix},$$

and

$$|i'_\perp\rangle \equiv \begin{bmatrix} \alpha'^*_y \\ -\alpha'^*_x \end{bmatrix},$$

where $|\alpha'_x|^2 + |\alpha'_y|^2 = 1$, then we will get outcome $i'$ with probability

$$\Pr(i' \mid |i\rangle) = |\langle i' | i \rangle|^2,$$

and outcome $i'_\perp$ with probability

$$\Pr(i'_\perp \mid |i\rangle) = |\langle i'_\perp | i \rangle|^2 = 1 - \Pr(i' \mid |i\rangle).$$

(a) Verify that the density operator for this pure state,

$$\hat{\rho} = |i\rangle\langle i|$$

gives these same probabilities via

$$\Pr(i' \mid |i\rangle) = \langle i' | \hat{\rho} | i' \rangle,$$

and

$$\Pr(i'_\perp \mid |i\rangle) = \langle i'_\perp | \hat{\rho} | i'_\perp \rangle = 1 - \Pr(i' \mid |i\rangle).$$

(b) Now suppose that the single photon is in a mixed state, i.e., that $\alpha_x$ and $\alpha_y$ are complex-valued random variables whose joint distribution ensures that $|\alpha_x|^2 + |\alpha_y|^2 = 1$ with probability one. Show that the density operator $\hat{\rho}$ can now be written in the form

$$\hat{\rho} = \begin{bmatrix} \langle |\alpha_x|^2 \rangle & \langle \alpha_x \alpha_y^* \rangle \\ \langle \alpha^*_x \alpha_y \rangle & \langle |\alpha_y|^2 \rangle \end{bmatrix},$$

by verifying that this expression yields the proper formulas for the unconditional measurement probabilities, $\Pr(i')$ and $\Pr(i'_\perp)$, i.e.,

$$\langle i' | \hat{\rho} | i' \rangle = \Pr(i') = \int d\alpha_x \int d\alpha_y p(\alpha_x, \alpha_y) \Pr(i' \mid |i\rangle),$$

and

$$\langle i'_\perp | \hat{\rho} | i'_\perp \rangle = \Pr(i'_\perp) = \int d\alpha_x \int d\alpha_y p(\alpha_x, \alpha_y) \Pr(i'_\perp \mid |i\rangle),$$

where $p(\alpha_x, \alpha_y)$ is the joint probability density for $\alpha_x$ and $\alpha_y$. Note that you have just shown that the preceding form of the density operator is equivalent to the mixed-state Poincaré vector that you studied in Problem 3.1.
Problem 3.5
Let \( \hat{A} \) and \( \hat{B} \) be observables for some quantum system. In particular, let \( \hat{A} \) and \( \hat{B} \) each be Hermitian operators with complete orthonormal (CON) sets of eigenkets,

\[
\{ |a_n\rangle : 1 \leq n < \infty \}
\]

and

\[
\{ |b_n\rangle : 1 \leq n < \infty \},
\]

and associated eigenvalues,

\[
\{ a_n : 1 \leq n < \infty \}
\]

and

\[
\{ b_n : 1 \leq n < \infty \},
\]

respectively.

(a) The commutator of \( \hat{A} \) and \( \hat{B} \) is, by definition,

\[
[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}.
\]

Show that \( \frac{1}{\mathcal{J}}[\hat{A}, \hat{B}] \) is an Hermitian operator.

(b) Assume that these observables commute, i.e.,

\[
[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = 0,
\]

and that the eigenvalues of \( \hat{A} \) are distinct, as are the eigenvalues of \( \hat{B} \). Show that every eigenket of \( \hat{A} \) is also an eigenket of \( \hat{B} \) and that every eigenket of \( \hat{B} \) is also an eigenket of \( \hat{A} \), i.e., \( \hat{A} \) and \( \hat{B} \) have a common, CON set of eigenkets.

Problem 3.6
Here we introduce the notation of tensor products, to permit us to deal with multiple quantum systems. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be the Hilbert spaces of possible states for two quantum systems, \( S_1 \) and \( S_2 \), respectively. If we are interested in making a joint measurement on these two systems, e.g., the sum of their “positions”, etc., we need to have a way to describe states and observables for the joint system. Let \( \{ |\phi_n\rangle_1 : 1 \leq n < \infty \} \) and \( \{ |\theta_m\rangle_2 : 1 \leq m < \infty \} \) be orthonormal bases for \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively, where the subscripts 1 and 2 indicate to which Hilbert space the states belong. The Hilbert space of states for the joint quantum system—i.e., systems 1 and 2 together—is spanned by the tensor product states \( \{ |\phi_n\rangle_1 \otimes |\theta_m\rangle_2 : 1 \leq n, m < \infty \} \), i.e., an arbitrary state \( |\psi\rangle \in \mathcal{H} \) can be expressed as a linear combination,

\[
|\psi\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} (|\phi_n\rangle_1 \otimes |\theta_m\rangle_2),
\]

by appropriate choice of the coefficients \( \{c_{nm}\} \). Thus, because the inner product between \( |\phi_n\rangle_1 \otimes |\theta_m\rangle_2 \) and \( |\phi_k\rangle_1 \otimes |\theta_l\rangle_2 \) is defined to be,

\[
(\langle \phi_l | \otimes \langle \theta_k |)(|\phi_n\rangle_1 \otimes |\theta_m\rangle_2) = (\langle \phi_l |\theta_m\rangle_2)(\langle \phi_k |\phi_n\rangle_1),
\]

the inner product between \( |\psi\rangle \) from Eq. (3) and

\[
|\psi'\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm} (|\phi_n\rangle_1 \otimes |\theta_m\rangle_2),
\]
is
\[ \langle \psi' | \psi \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm}^* c_{mn}. \]

(a) Let \( \hat{A}_1 \) be an observable of system 1, i.e., an Hermitian operator on \( \mathcal{H}_1 \) with a complete set of eigenkets, and let \( \hat{B}_2 \) be an observable of system 2, i.e., an Hermitian operator on \( \mathcal{H}_2 \) with a complete set of eigenkets. The tensor product \( \hat{C} = \hat{A}_1 \otimes \hat{B}_2 \) is a linear operator that maps the state \( |\phi_n\rangle_1 \otimes |\theta_m\rangle_2 \) into the state \( (\hat{A}_1 |\phi_n\rangle_1) \otimes (\hat{B}_2 |\theta_m\rangle_2) \).

Show that \( \hat{C} \) is an Hermitian operator on \( \mathcal{H} \) which has a complete set of eigenkets, so that \( \hat{C} \) is an observable on the joint Hilbert space of systems 1 and 2.

(b) Let
\[ \hat{A}_1 = \sum_{n=1}^{\infty} a_n |a_n\rangle_{11} \langle a_n| \quad \text{and} \quad \hat{B}_2 = \sum_{m=1}^{\infty} b_m |b_m\rangle_{22} \langle b_m| \]
be the diagonal (eigenvalue/eigenket) decompositions of \( \hat{A}_1 \) and \( \hat{B}_2 \), where the \( \{a_n\} \) are assumed to be distinct, as are the \( \{b_m\} \). When we measure \( \hat{A}_1 \) on system 1 and we measure \( \hat{B}_2 \) on system 2 with the joint system being in state \( |\psi\rangle \), given by Eq. (3), the outcome will be an ordered pair \( \{(a_n, b_m)\} \) of eigenvalues. The probability that \( (a_n, b_m) \) occurs is given by,
\[ \Pr(a_n, b_m) = |\langle \psi ||a_n\rangle_1 \otimes |b_m\rangle_2 \rangle|^2. \]

Show that this is a proper probability distribution. Express the marginal probabilities, \( \Pr(a_n) \) and \( \Pr(b_m) \), in terms of \( |\psi\rangle \), the \( \{|a_n\rangle_1\} \) and the \( \{|b_m\rangle_2\} \).

(c) Specialize the results of (b) to the case of a product state, viz., a state that satisfies \( |\psi\rangle = |\psi_1\rangle_1 \otimes |\psi_2\rangle_2 \).

Problem 3.7
Here we prove that it is impossible to clone the unknown state of a quantum system by means of a unitary evolution. It is a proof by contradiction. Suppose that we have a quantum system whose Hilbert space of states is \( \mathcal{H}_S \), where \( S \) indicates that this is the source system. Suppose too that we have a target system whose Hilbert space of states is \( \mathcal{H}_T \). We will assume that these two Hilbert spaces have the same dimensionality, e.g., 2.

We wish to construct a perfect cloner, viz., a unitary operator, \( \hat{U} \), on the tensor product space \( \mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_T \) such that
\[ \hat{U} (|\psi\rangle_S \otimes |0\rangle_T) = |\psi\rangle_S \otimes |\psi\rangle_T, \quad (4) \]
where $|\psi\rangle_S$ is an arbitrary unit-length ket in $\mathcal{H}_S$, and $|0\rangle_T$ is a reference ("blank") unit-length ket in $\mathcal{H}_T$. Thus, the perfect cloner does not disturb the source state while it turns the target’s “blank” state into a clone of the source state.

Let $|\psi_1\rangle_S$ and $|\psi_2\rangle_S$ be two distinct, unit-length kets in $\mathcal{H}_S$, let $\alpha$ and $\beta$ be two non-zero complex numbers, and assume that we have found a perfect cloner operator $\hat{U}$ that satisfies Eq. (4) for all unit-length source kets.

(a) Define

$$|\psi'\rangle_S = \frac{\alpha|\psi_1\rangle_S + \beta|\psi_2\rangle_S}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta(\langle\psi_1|\psi_2\rangle_S)]}}.$$ 

Use unitarity to evaluate the length of the ket $|\theta\rangle \equiv \hat{U}(|\psi'\rangle_S \otimes |0\rangle_T)$.

(b) Use the linearity of $\hat{U}$ to show that

$$|\theta\rangle = \alpha'(|\psi_1\rangle_S \otimes |\psi_1\rangle_T) + \beta'(|\psi_2\rangle_s \otimes |\psi_2\rangle_T),$$

where

$$\alpha' \equiv \frac{\alpha}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta(\langle\psi_1|\psi_2\rangle_S)]}},$$

$$\beta' \equiv \frac{\beta}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta(\langle\psi_1|\psi_2\rangle_S)]}}.$$ 

(c) Use Eq. (5) to evaluate the length of $|\theta\rangle$. Show that this result contradicts what you found in (a), and thus conclude that there is no unitary $\hat{U}$ that can be a perfect cloner.

**Problem 3.8**

Here we prove that it is impossible to erase the unknown state of a quantum system by means of a unitary evolution. It is a proof by contradiction. Suppose that we have a quantum system whose Hilbert space of states is $\mathcal{H}_S$, where $S$ indicates that this is the source system. Suppose too that we have an ancilla system whose Hilbert space of states is $\mathcal{H}_A$. We will assume that these two Hilbert spaces have the same dimensionality, e.g., 2.

We wish to construct a perfect eraser, viz., a unitary operator, $\hat{U}$, on the tensor product space $\mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_A$ such that

$$\hat{U}(|\psi\rangle_S \otimes |0\rangle_A) = |0\rangle_S \otimes |0\rangle_A,$$

where $|\psi\rangle_S$ is an arbitrary unit-length ket in $\mathcal{H}_S$, and $|0\rangle_A$ is a reference ("blank") unit-length ket in $\mathcal{H}_A$. Thus, the perfect eraser does not disturb the ancilla state while it turns the source’s input state into a “blank.”

Let $|\psi_1\rangle_S$ and $|\psi_2\rangle_S$ be two distinct, unit-length kets in $\mathcal{H}_S$, let $\alpha$ and $\beta$ be two non-zero complex numbers, and assume that we have found a perfect eraser operator $\hat{U}$ that satisfies Eq. (6) for all unit-length source kets.
(a) Define
\[ |\psi'\rangle_S = \frac{\alpha |\psi_1\rangle_S + \beta |\psi_2\rangle_S}{\sqrt{\alpha^2 + \beta^2 + 2\text{Re}[\alpha^* \beta (S\langle \psi_1 | \psi_2 \rangle_S)]}}. \]

Use unitarity to evaluate the length of the ket \( |\theta\rangle \equiv \hat{U}(|\psi'\rangle_S \otimes |0\rangle_A). \)

(b) Use the linearity of \( \hat{U} \) to show that
\[ |\theta\rangle = (\alpha' + \beta') (|0\rangle_S \otimes |0\rangle_A). \quad (7) \]

where
\[ \alpha' \equiv \frac{\alpha}{\sqrt{\alpha^2 + \beta^2 + 2\text{Re}[\alpha^* \beta (S\langle \psi_1 | \psi_2 \rangle_S)]}} \]
\[ \beta' \equiv \frac{\beta}{\sqrt{\alpha^2 + \beta^2 + 2\text{Re}[\alpha^* \beta (S\langle \psi_1 | \psi_2 \rangle_S)]}} \]

(c) Use Eq. (7) to evaluate the length of \( |\theta\rangle \). Show that this result contradicts what you found in (a), and thus conclude that there is no unitary \( \hat{U} \) that can be a perfect eraser.