Announcements

- Pick up lecture notes, slides

Two-Mode Linear Systems

- Parametric amplifier statistics
- Entanglement

Four-Mode Quantum Systems

- Polarization entanglement
- Qubit teleportation
**Parametric Amplifier with Gain $G$**

\[ \hat{a}_{\text{out}_x} = \sqrt{G} \hat{a}_{\text{in}_x} + \sqrt{G-1} \hat{a}_{\text{in}_y}^\dagger \]

\[ \hat{a}_{\text{out}_y} = \sqrt{G} \hat{a}_{\text{in}_y} + \sqrt{G-1} \hat{a}_{\text{in}_x}^\dagger \]

- Two-Mode Bogoliubov Transformation:

**Output State of the Parametric Amplifier**

- Quantum Characteristic-Function Analysis:

\[ \chi^\rho_{\text{out}}(\xi_x^*, \xi_y^*, \xi_x, \xi_y) \equiv \langle e^{-\xi_x^* \hat{a}_{\text{out}_x} - \xi_x \hat{a}_{\text{out}_x}^\dagger + \xi_y \hat{a}_{\text{out}_y}^\dagger - \xi_y^* \hat{a}_{\text{out}_y}} \rangle \]

\[ = \chi^\rho_{\text{in}}(\xi_x^*, \xi_y^*, \xi_x, \xi_y) \]

\[ \xi_x \equiv \sqrt{G} \xi_x - \sqrt{G-1} \xi_y^* \text{ and } \xi_y \equiv \sqrt{G} \xi_y - \sqrt{G-1} \xi_x^* \]

- Important Special Case: Vacuum-State Inputs

\[ \chi^\rho_{\text{out}}(\xi_x^*, \xi_y^*, \xi_x, \xi_y) = e^{-G(|\xi_x|^2+|\xi_y|^2)+2\sqrt{G(G-1)}\Re(\xi_x\xi_y^*)} \]

\[ \neq \chi^\rho_{\text{out}_x}(\xi_x^*, \xi_x) \chi^\rho_{\text{out}_y}(\xi_y^*, \xi_y) \]

output state is entangled
Parametric Amplifier Output with Vacuum Inputs

- Individual Output Modes are in Classical States:
  \[ \chi_{\alpha}^{A} (\zeta_x, \zeta_x) = \chi_{\alpha}^{A} (\zeta_x, 0, 0) = e^{-G|\zeta_x|^2} \]
  \[ = \langle 0 | e^{-\zeta_x^* \hat{a}_{out_x} e^{\zeta_x \hat{a}_{out_x}^t} |0 \rangle e^{-(G-1)|\zeta_x|^2} \]
  \[ P_{out_x} (\alpha, \alpha^*) = P_{out_y} (\alpha, \alpha^*) = \frac{e^{-|\alpha|^2/(G-1)}}{\pi (G-1)} \]

- Each output is a phase-insensitive-amplified zero-input field

- Joint Output State is Non-Classical:
  - The ±45° (diagonal) basis modes are in squeezed vacuum states

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- Squeezed-Vacuum Representation:
  \[ |\psi\rangle_{out} = |\psi\rangle_{out_{45}} |\psi\rangle_{out_{-45}} \]
  \[ = |0; \sqrt{G}, -\sqrt{G-1}\rangle_{45} |0; \sqrt{G}, \sqrt{G-1}\rangle_{-45} \]

- Photon-Twins Representation:
  \[ |\psi\rangle_{out} = \sum_{n=0}^{\infty} \sqrt{\frac{(G-1)^n}{G^{n+1}}} |n\rangle_x |n\rangle_y \]
Photon-Twins Behavior

- Semiclassical Theory for Deterministic $a$ :
  \[ \langle \Delta N^2 \rangle = \langle N_x \rangle + \langle N_y \rangle \]

- Quantum Theory for Paramp Outputs $\hat{a}$ from Vacuum Inputs:
  \[ \langle \Delta \hat{N}^2 \rangle = 0 \]

Reduced Density Operators for Paramp Outputs

- Measure Observable $\hat{O}_x = \sum_n o_n |o_n\rangle_x x \langle o_n|$ :
  \[ \text{Pr}(\hat{O}_x = o_n | \psi_{\text{out}}) = \text{tr}[\hat{\rho}_{\text{out}}(|o_n\rangle_x x \langle o_n| \otimes \hat{I}_y)] \]
  \[ = \text{tr}(\hat{\rho}_{\text{out}_x} |o_n\rangle_x x \langle o_n|) \]
  implies $\hat{\rho}_{\text{out}_x} = \text{tr}_y(\hat{\rho}_{\text{out}})$

- For the Parametric Amplifier with Vacuum-State Inputs
  \[ \hat{\rho}_{\text{out}_k} = \sum_{n=0}^{\infty} \frac{(G - 1)^n}{G^{n+1}} |n\rangle_k k \langle n|, \text{ for } k = x, y \]
Creating Polarization-Entangled Photon Pairs

- Polarization-Combined Outputs from Anti-Phased Paramps

\[ \hat{a}_{Bx} i_x + \hat{a}_{Ay} i_y \quad \text{Output 2} \]

\[ \hat{a}_{Ax} i_x \rightarrow \hat{a}_{Ay} i_y \quad \text{Output 1} \]

- Low-gain operation: \( 0 < G - 1 = \Delta G \ll 1 \)

Creating Polarization-Entangled Photon Pairs

- Dual-Paramp Output State in Low-Gain Limit:

\[ |\psi\rangle_{\text{out}} = |\psi\rangle_A \otimes |\psi\rangle_B \]

\[ = \sum_{n=0}^{\infty} \sqrt{\frac{\Delta G^n}{G^{n+1}}} |n\rangle_A |n\rangle_A \otimes \sum_{m=0}^{\infty} (-1)^m \sqrt{\frac{\Delta G^m}{G^{m+1}}} |m\rangle_B |m\rangle_B \]

\[ \approx (|0\rangle_A |0\rangle_B) \otimes (|0\rangle_B |0\rangle_A) \]

\[ + \sqrt{\Delta G} \left[ (|1\rangle_A |0\rangle_B) \otimes (|0\rangle_B |1\rangle_A) - (|0\rangle_A |1\rangle_B) \otimes (|1\rangle_B |0\rangle_A) \right] \]

- Entangled Bi-Photon State Realized by Post-Selection:

\[ \frac{1}{\sqrt{2}} (|x\rangle_1 |y\rangle_2 - |y\rangle_1 |x\rangle_2) \]
**Polarization-Entangled Photon Pairs**

- Measure Outputs 1 and 2 in Conjugate-Pair Polarizations:
  \[ |i\rangle_k = \alpha |x\rangle_k + \beta |y\rangle_k \text{ and } |i'\rangle_k = \beta^* |x\rangle_k - \alpha^* |y\rangle_k, \text{ for } k = 1, 2 \]

- Polarization Entanglement: \( \Pr (N_{2i} = 1 \mid N_{1i} = 1) = 1 \)

**The Four Steps of Qubit Teleportation**

- **Step 1**: Entanglement Source
- **Step 2**: Message, Transmitter, Joint Measurement
- **Step 3**: Classical Communication, State Transformation
- **Step 4**: Receiver, Teleported Message
Coming Attractions: Lecture 14

- Lecture 14: Teleportation
  - Polarization entanglement and qubit teleportation
  - Quadrature entanglement and continuous-variable teleportation
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