## LECTURE 1: The Quantification of Sound and the Wave Equation

*Required Reading:* Denes & Pinson Chapters 1-3;
Supplemental reading: Kinsler et al (KFCS) pg 99-111

### 1. The Basic Physical Attributes of Sound: What is Sound? How is it Produced? How does it Propagate?

Sound is a propagating mechanical disturbance in a medium. Propagation of sound does not occur via net translocation of matter, it is the mechanical disturbance that propagates. Most of the course will be spent discussing sound in fluid media such as air and water. The picture on the right shows a cam driven piston at the end of a rigid tube. When the piston is set into oscillation, successive layers of the air medium within the tube are also set into motion causing local increases and decreases in the density of air (coded in shades of gray). Associated with the net back and forth oscillations of the air particles are increases (condensations) and decreases (rarefactions) in local pressure. The velocity with which the particle oscillates is the particle velocity, while the velocity with which the disturbance in pressure and velocity moves down the tube is the propagation velocity of sound. Note that the variations in density have been greatly amplified for viewing purposes (after HF Olsen, 'Music Physics and Engineering' Dover Press 1967).

Image removed due to copyright considerations.

Source: Olsen, H. F. *Music Physics and Engineering.* Dover Press, 1967.

*A. Sound Pressure, p(t),* is the variation about the baseline pressure that results from the alternating condensations and rarefactions of media that describe the propagating sound wave. The units of sound pressure are *pascals*, where 1 Pa = 1 newton/m$^2$. A sound pressure of 1 Pa at 1000 Hz is of uncomfortable but not painful loudness. This loud pressure is equivalent to 1/100,000 of an atmosphere and 50,000 times the lowest sound pressures that are audible. *Sound Pressure* is a scalar quantity.

*B. Particle Velocity,* $\bar{v}(t)$ is a vector quantity that describes the alternating average velocity of motion of a particle of medium. The units of particle velocity are m/s.

An *acoustic particle* is "a volume element large enough to contain millions of molecules so that the fluid may be thought of as a continuous medium, yet small enough that all acoustic variables may be considered nearly constant throughout..." (KFC&S, page 99).
Particle size depends on the medium and the frequency. A medium excited by large wave length sounds can be broken into larger (more voluminous) particles than a medium excited by sounds with

smaller wave lengths. What's important is that within the particle the sound pressure and the average motion of the particles is constant. A particle of free air exposed to a sinusoidal sound pressure of 1 Pa, moves back and forth with a velocity amplitude of about 2 mm/s. *How does that compare with the propagation velocity (speed of sound) in air?*

The definition of an acoustic particle is relevant to an important issue in acoustics, i.e. whether we can consider a system to be made up of a collection of '*lumped elements*' which behave as particles or '*distributed systems*' which behave like continuous media and support wave motion. We will revisit this distinction regularly throughout the coming term.

*C. Scalars and Vectors*

*Scalars* describe nondirectional physical processes like *pressure*. *Vectors* describe the magnitude and direction of directional physical processes like *force* and *velocity.* A vector can be broken into its three-dimensional components, e.g.

$$\bar{v} = \bar{i}_x v_x + \bar{i}_y v_y + \bar{i}_z v_z \qquad (1.0)$$

Where: $\bar{i}_x, \bar{i}_y$ *and* $\bar{i}_z$ are unit vectors in the *x, y and z* directions, and

$v_x, v_y$ *and* $v_z$ are scalars that define the magnitudes of the *x, y and z* component vectors.

*D. Sound Frequency, f,* describes the temporal variation of a pure tone, e.g.

$$p(t) = A\cos(2\pi f\, t + \theta), \text{ where } 2\pi f = \omega, \text{ the radian frequency} \qquad (1.1)$$

*E. The Density of the Sound Conducting Medium,* ρ, is the mass per unit volume of the medium with SI units of kg/m$^3$. The density of air at Standard Temperature & Pressure (20°C and 1 atm) $\rho_0$ is about 1.21 kg/m$^3$. The density of gases goes up as pressure increases and goes down as temperature increases.

*F. Linear Acoustics*

For most, if not all, of this course we will deal with sound flow through fluid media (mostly air) in a regimen know as linear acoustics. As sound travels through a medium there are temporal and spatial variations in the pressure, density, particle velocity and temperature associated with the sound.

sound pressure: $p(x,y,z,t) = p_{Total}(x,y,z,t) - P_0$,

particle velocity: $\bar{v}(x,y,z,t) = v_x(x,y,z,t)\bar{i}_x + v_y(...)\bar{i}_y + v_z(...)\bar{i}_z$,

sound density: $\rho(x,y,z,t) = \rho_{Total}(x,y,z,t) - \rho_0$ and

sound temperature: $T(x,y,z,t) = T_{Total}(x,y,z,t) - T_0$

In linear acoustics the sound-induced variations in pressure *p*, density ρ and temperature *T* are small compared to the baseline value of these quantities $P_0$, $\rho_0$ and $T_0$.

*G. The Bulk modulus, B* of a material is the pressure difference associated with a fractional change in the volume of the material. If we consider a collection of air particles of volume $V_1$ at pressure $P_1$, and we then change the pressure acting on the volume to $P_2$ with a resultant new volume of $V_2$, the bulk modulus is:

$$B = \frac{P_2 - P_1}{(V_1 - V_2)/V_2} \quad \text{with units of pressure.} \tag{1.2}$$

For ideal gases at 1 atmosphere with <u>isothermal compression</u> Boyles' law tells us $B$=1 atm =$10^5$ Pa. In cases of <u>adiabatic compression</u> of an ideal gas at $P_0$ = 1 atm: $B_A = \gamma P_0 = \gamma\ 10^5$ Pa. For diatomic gases like $N_2$ and $O_2$ γ, the ratio of specific heats, = 1.4 . For monatomic gases like He γ = 1.67 . Air, which is mostly diatomic gases has a specific heat of γ = 1.41 such that at 1 atm and 20° C, $B_A$= 1.41 × $10^5$ Pa .

*<u>H. The Propagation Velocity of Sound</u>, c,* depends on the stiffness and density of the sound-conducting medium, i.e.

$$c = \sqrt{B_A/\rho_0} \quad \text{with units of m/s.} \tag{1.3}$$

*Why do we use the adiabatic compressibility?*

At standard temperature and pressure (20°C and 1 atm) <u>$c \approx 340$ m/s</u>.

*<u>I. The wavelength of sound</u> λ,* depends on sound frequency and the propagation velocity,

$$\lambda = c/f \quad \text{with units of meters per cycle} \tag{1.4}$$

Since the human ear is sensitive to sound pressures of frequencies varying from 20 Hz to 20,000 Hz (aka 20 kHz), the wavelengths of the sounds to which we are sensitive vary from over 10 meters to less than 2 cm. This large variation from wave lengths that are much bigger than the objects around us to wave lengths that are smaller than the objects around us is one of the challenges of acoustics.

| Frequency (Hz) | Wave length in air | Comparable Structure | Wave length in water | Comparable Structure |
|---|---|---|---|---|
| 31.5 | 10.8 m | class room | 47m | Olympic pool |
| 100 | 3.4 | bed room | 15 | small yacht |
| 315 | 1.08 | torso & head | 4.7 | small boat |
| 1,000 | 0.34 | head | 1.5 | small human |
| 3,150 | 0.108 | vocal tract | 0.47 | tuna |
| 10,000 | 0.034 | ear canal | 0.15 | mackerel |
| 31,500 | 0.0108 | human TM | 0.047 | anchovy |
| 100,000 | 0.0034 | ossicle/gnat | 0.015 | chum |

*<u>J. The Characteristic Impedance</u>, $z_0$,* is another property of the sound conducting medium that depends on the stiffness and density of the medium, i.e.

$$z_0 = \sqrt{B_A \rho_0} = \rho_0 c\,. \tag{1.5}$$

The characteristic impedance of a media relates the sound induced variations in pressure and particle velocity. <u>For the special case of a plane wave propagating in the *x* direction in free open space</u>

$$p(t) = z_0 v_x(t). \qquad (1.6)$$

The unit of characteristic impedance is the rayl, named after Lord Rayleigh, where 1 rayl = 1Pa-s/m, i.e. the ratio of a pressure and a velocity.

How does the characteristic impedance compare with electrical impedance and mechanical impedance?

**All of the formulae we have introduced thus far contain only *real numbers*.** This simplification is consistent with the plane-wave open-space constraint on Eqn 1.6, where that equation specifies that the patterns of temporal variations in particle velocity and pressure are identical. As we will soon see, there can be significant differences in the temporal patterns of velocity and pressure in sound resulting in descriptions of impedance that depend on complex numbers.

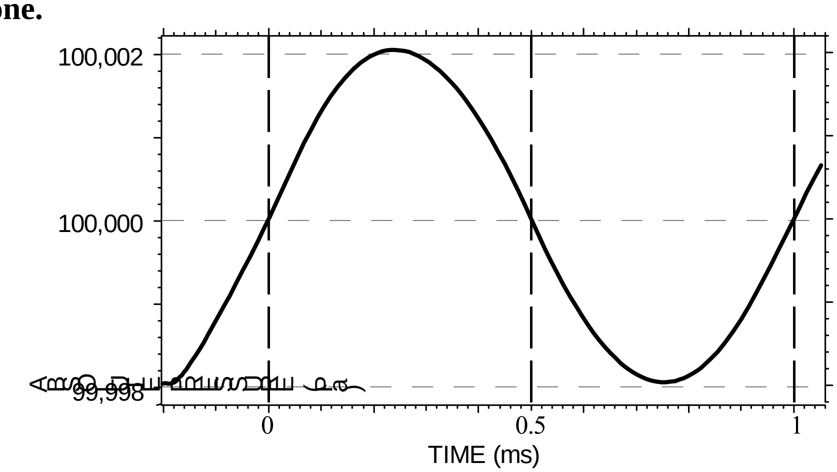
**2. Simple Sounds: the Pure Tone.**

Figure 1.2

The *total pressure* as a function of time is

$$p_T(t)=P_0+p(t),$$

at a point $x$ in space in the presence of a propagating plane wave produced by a sound source playing a continuous tone. The sound pressure is zero at time $t$ =... 0, 0.5, 1 ms ...




What is the frequency of the tone in Fig 1.2?

What is the peak amplitude of the sound pressure in Fig 1.2?

What is the root-mean-square or *rms* amplitude of the sound pressure in Fig. 1.2?

To compute *rms* amplitude: square the waveform, average it over some characteristic time, and then take the square root of the average.

Describe the wave in Figure 1.2 in terms of a cosine function of time.

What is the amplitude/magnitude of the cosine function?

What is the frequency?

What is the phase angle?

*A. Magnitude and Phase: Complex notation*

The fact that sinusoids can vary in magnitude and phase can be coded in terms of a complex number, e.g. $a + jb$, where $a$ is the real part of the number, $b$ is the imaginary part of the number and

$j$ = the imaginary number $\sqrt{-1}$ . I use italic script to represent variables where lower-case variables generally refer to real constants or functions of time and upper-case variables generally refer to quantities that vary with frequency. (An exception to this rule is $B_A$.) I denote complex variables with an underbar, e.g. $\underline{B} = a + jb$.

We can convert between the "rectangular" notion for complex numbers $\underline{B} = a + jb$, and a "polar" notation $\underline{B} = |\underline{B}|e^{j\angle\underline{B}}$ where the magnitude of the complex number $\underline{B}$ is

$$|\underline{B}| = \sqrt{a^2 + b^2}, \qquad (1.7)$$

and the angle of the complex number $\underline{B}$ is

$$\angle\underline{B} = atan(b/a). \qquad (1.8)$$

We can convert a complex magnitude and angle back into the rectangular real and imaginary components using *Euler's Equations*, such that for $\underline{B} = a + jb$:

$$\begin{aligned}\text{Real}\{\underline{B}\} &= a = |\underline{B}|\cos(\angle\underline{B}), \text{ and}\\ \text{Imaginary}\{\underline{B}\} &= b = |\underline{B}|\sin(\angle\underline{B})\end{aligned} \qquad (1.)$$

*B. Euler's Equations: the relationship between complex exponentials and sinusoids.*

| | |
|---|---|
| i. Euler's equations: where $j = \sqrt{-1}$ | $e^{j\theta} = \cos\theta + j\sin\theta$ <br> $e^{-j\theta} = \cos\theta - j\sin\theta$ |
| ii. A specific example: | $Ae^{j(\omega t+\phi)} = A(\cos(\omega t + \phi) + j\sin(\omega t + \phi))$ |
| iii. Separation of (ii) into two complex amplitudes, one that is constant and one dependent on time | $Ae^{j\phi}e^{j\omega t} = A(\cos(\omega t + \phi) + j\sin(\omega t + \phi))$ |
| iv. A complex amplitude $\underline{B}$ with magnitude $\|\underline{B}\|$ and phase angle $\angle\underline{B}$ | $\underline{B}e^{j\omega t} = \|\underline{B}\|(\cos(\omega t + \angle\underline{B}) + j\sin(\omega t + \angle\underline{B}))$ |
| v. Description of a simple cosine function by a complex exponential | $\text{Real}\{\underline{B}e^{j\omega t}\} = \|\underline{B}\|\cos(\omega t + \angle\underline{B})$ |

Row (i) notes Euler's basic equations where a complex exponential amplitude of magnitude 1 and angle θ noted as $e^{j\theta}$ has a real part of cosθ and an imaginary part of sinθ.

Row (ii) gives another example of a complex exponential of magnitude = $A$ and angle = ($\omega t+\phi$). Note that the angle argument has a component that varies with time $t$ and *radian frequency* $\omega=2\pi f$, and a component that is constant $\phi$.

In Row (iii) the complex exponential from the previous row is split into two components by simple algebra, i.e. $Ae^{j(\omega t+\phi)} = Ae^{j\phi}Ae^{j\omega t}$.

In Row (iv) we see the combination of two complex numbers $\underline{B} = |\underline{B}|e^{j\angle\underline{B}}$ and $e^{j\omega t}$ yielding an argument to the cosine and sine terms made up of the time varying $\omega t$ and the constant $\angle\underline{B}$.

Row (v) describes a cosine function of magnitude $|\underline{B}|$ and angle $\angle\underline{B}$ as the real part of the product of two complex exponentials, one constant and the other time dependent.

*C. The Specific Acoustic Impedance $\underline{Z}_S$*

With the introduction of complex amplitudes we can now give a more general description of acoustic impedance. Suppose a sinusoidal source of frequency *f* produces a sound wave that passes through a point *Q* while propagating in direction *x*. We can define the temporal variations in the sound pressure and the *x*-component of the velocity at point *Q* in terms of complex exponential amplitudes, where $\omega=2\pi f$, and

$$p(t) = |\underline{P}|\cos(\omega t + \angle\underline{P}) = \mathrm{Real}\{\underline{P}e^{j\omega t}\}, \text{ and}$$

$$\bar{v}_x(t) = |\underline{V}|\cos(\omega t + \angle\underline{V}) = \mathrm{Real}\{\underline{V}e^{j\omega t}\}. \quad (1.10\&b)$$

The specific acoustic impedance relating sound pressure and particle velocity at point *Q* is defined by the ratio of the complex amplitudes $\underline{P}$ and $\underline{V}$, i.e.

$$\underline{Z}^S = \underline{P}/\underline{V} \; . \quad (1.11)$$

Note that $\underline{Z}_S$ is:

-complex (it has a magnitude and an angle),
-independent of time,
-like the characteristic impedance of the medium has units of rayls, and
-unlike the characteristic impedance that describes the propagation of sound in an infinite expanse of a medium, the specific acoustic impedance can describe relationships between *p(t)* and *v(t)* where the two time functions are out of phase. This is important when describing sound near objects, i.e. not in the free-field.

*D. Sound Intensity and Power.*

A more general metric of the amplitude of a sound wave is the *Intensity* or energy per unit time per unit area (joule/s/m$^2$=watt/m$^2$). This quantity is sometimes called power density. Why?

The *average intensity* of a sound wave is a real quantity related to the product of the sound pressure and the particle velocity. We can define the *instantaneous intensity* in the *x* direction:

$$i_x(t)=p(t)\, v_x(t). \quad (1.12)$$

In the sinusoidal steady state:

$$i_x(t) = |\underline{P}|\cos(\omega t + \angle\underline{P})\, |\underline{V}|\cos(\omega t + \angle\underline{V}). \quad (1.13)$$

Using the identity

$$\cos a \cos b = \frac{\cos(a+b)+\cos(a-b)}{2}, \text{ leads to}$$

$$i_x(t) = \frac{|\underline{P}||\underline{V}|}{2}\cos(2\omega t + \angle\underline{P} + \angle\underline{V}) + \frac{|\underline{P}||\underline{V}|}{2}\cos(\angle\underline{P} - \angle\underline{V}). \quad (1.14)$$

Since we are really interested in the intensity averaged over some time, we only need consider part of Eqn 1.14. In particular note that the first term on the right-side of (1.14) is a cosine function with a frequency that is twice the frequency of the pressure and velocity variation. The temporal average of such a sinusoidal function is zero for each cycle, and that term does not contribute to the *average intensity* which is a constant completely defined by the second term on the right,

$$\bar{I} = \frac{|\underline{P}||\underline{V}|}{2}\cos(\angle\underline{P} - \angle\underline{V}). \quad (1.15a)$$

Note that the *average intensity* depends greatly on the phase relationship between $\underline{P}$ and $\underline{V}$. If $\angle\underline{P}=\angle\underline{V}$ then $\bar{I}=|\underline{PV}|/2$. If $\angle P$ and $\angle V$ differ by $\pi/2$ then $\bar{I}=0$.

(1.15a) can also be written in terms of complex exponentials as

$$\bar{I} = \frac{1}{2}\text{Real}\left\{\underline{P}\,\underline{V}^*\right\}, \tag{1.15b}$$

where $\underline{V}^*$ is the complex conjugate of $\underline{V}$, i.e. $\underline{V}^* = |\underline{V}|\, e^{-j\angle \underline{V}}$.

We can also use the Specific Acoustic Impedance relating $\underline{P}$ and $\underline{V}$ to define sound intensity, where: $\underline{Z}^S = \underline{P}/\underline{V}$. Substituting this relationship into 1.15b and realizing that $\underline{V}\,\underline{V}^* = |\underline{V}|^2$ yields:

$$\bar{I} = \frac{1}{2}\text{Real}\left\{\underline{P}\,\underline{V}^*\right\} = \frac{1}{2}|\underline{V}|^2\,\text{Real}\left\{\underline{Z}^S\right\} = \frac{1}{2}|\underline{P}|^2\,\text{Real}\left\{\frac{1}{\underline{Z}^S}\right\}. \tag{1.16}$$

In the case of a plane wave $\underline{Z}^S = z_0 = \rho_0 c$ and is real.

The average sound power $\overline{\Pi}$ is $\bar{I}$ (with units of power/area) times the area of the power collecting surface that is orthogonal to the direction of propagation of the wave.

*E. The decibel, dB, as Measure of Pressure and Intensity*.

The decibel is a logarithmic description of the ratio of two energy levels. The use of logarithmic scales to describe sound pressure and intensity stems from two facts:
(1) The human ear responds to a wide range of sound intensities (about 6 orders of magnitude) and humans generally like to describe quantities in as few significant units as possible. (2) Our ability to distinguish differences in the intensity of two sound signals, can be roughly described in terms of a threshold fractional difference in the intensity between the two, i.e. sounds of intensity that are different by more than 25% are generally distinguishable while sounds of closer relative intensity are not. This sensitivity to fractional changes is just what logarithmic scaling is all about, and is a common approximation for human sensation. (You'll hear more about Weber's and Fechner's laws of psychophysical detection later in this course.)

The bel is a unit named after Boston's own Alexander Graham Bell, where 1 bel describes an order of magnitude change in energy. The decibel or dB breaks the bel into ten pieces where in the case of sound intensity (which is proportional to energy):

$$\textit{The dB value of Intensity} = 10\log_{10}\frac{\bar{I}}{I_0}, \tag{1.17}$$

where $I_0$ is some arbitrary reference intensity level.
A common intensity reference level in Speech and Hearing Science is the *Sound Pressure Level* (*SPL*) reference of $10^{-12}$watts/m$^2$.

Equation 1.17 can be used to compute the dB level re some reference for any quantity that is proportional to energy, e.g. power or sound pressure. However intensity and power are proportional to the square of sound pressure (see 1.16), and therefore to convert pressure measurements to some dB term:

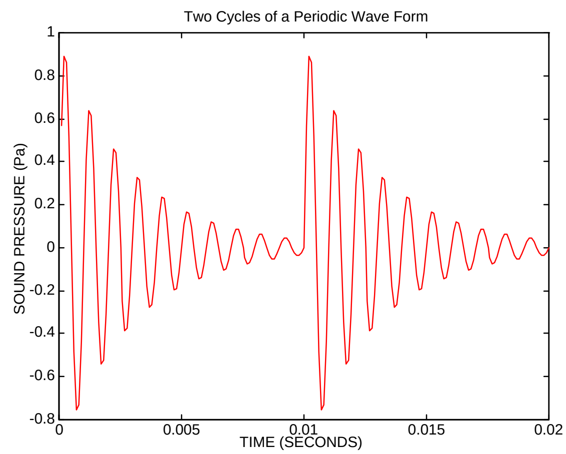
$$\textit{The dB value of Intensity} = 10\log_{10}\left(\frac{|\underline{P}|}{P_{Ref}}\right)^2 = 20\log_{10}\frac{|\underline{P}|}{P_{Ref}}\quad. \tag{1.18}$$

The sound pressure reference level for *Sound Pressure Level* is 2x10$^{-5}$ Pa. You should use equation 1.16 to convince yourself that the intensity and pressure references for *SPL* are consistent with each other.

## 3. Periodic Complex Signals

Figure 1.3 Shows two periods of a periodic complex acoustic signal that can be characterized as an 'exponentially decaying tone'. The period of each repetition is 10 ms, the period of the 'tonal' component of the signal is 1 ms.

The signal has an rms amplitude of 0.27 Pa. What is its peak amplitude?




What are the frequencies within the complex signal of Figure 1.3?

What is the repetition frequency?

What is the frequency of the tonal component?

Are there other frequencies?

*A. Fourier's Theorem applied to periodic signals*

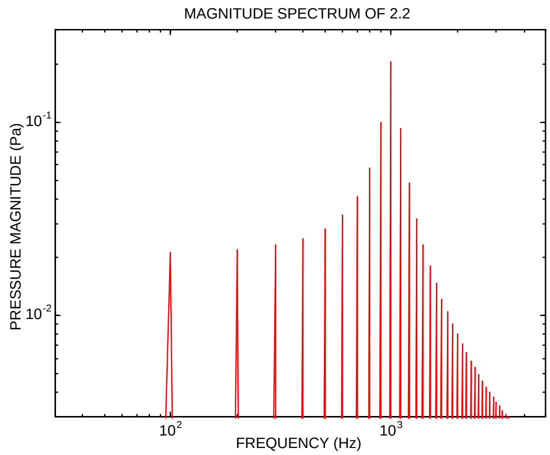
Any periodic signal of period $T$ can be reconstructed from the sum of a static component and a series of sinusoidal components that are harmonics of the repetition frequency. In general:

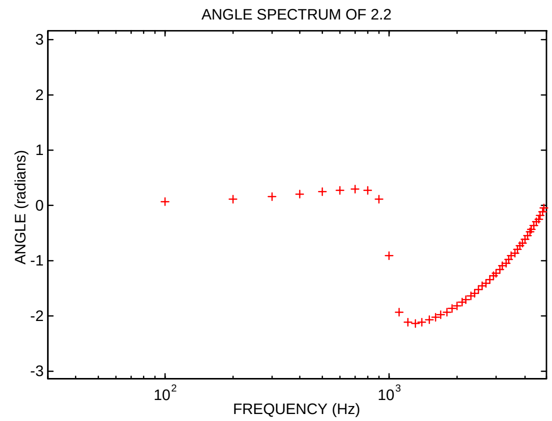
For $p(t)$ with period $T$,

$$p(t) = P_0 + \sum_{n=1}^{\infty} |\underline{P}_n| \cos\left(n 2\pi \frac{1}{T} + \angle \underline{P}_n\right). \quad (1.19)$$

Fourier analysis of the time wave form in Figure 1.3 yields the magnitude and angle spectra below, where the lines on the left show the magnitude of the complex Fourier components $\underline{P}_n$ and the pluses on the left show the angles of the same components.

Figure 1.4 Bode plots of the Magnitude and Angle of the Fourier Components that describe *p(t)* in Fig. 1.3



*B. Summing sound pressures of different frequencies*

The rms value of *p(t)* in Figure 1.3 can also be computed from the magnitude spectrum by summing the 'intensities' of the spectral components of Figure 1.4 components and converting back to pressure.

$$P_{rms}^2/\rho_0 c = \frac{1}{2}\Sigma\left(\left|\underline{P}_n\right|^2/\rho_0 c\right) \quad (1.20)$$

$$P_{rms} = \frac{1}{\sqrt{2}}\sqrt{\Sigma\left(\left|\underline{P}_n\right|^2\right)} = 0.27 \text{ Pa}$$

As a simpler example, suppose we have two sound sources one of 1000 Hz and another of 1500 Hz that each produce tonal amplitudes of 2 Pa at some point Q, such that at Q:

$$p_1(t) = 2\cos(2\pi 1000t), p_2(t) = 2\cos(2\pi 1500t)$$

The *rms* sound pressure when either source is active by itself is 1.41 Pa.
What is the *rms* sound pressure when both sources are turned on?

Since it is the intensity that adds: the total intensity is:

$$I_{Total} = \frac{1}{2}\frac{P_1^2}{z_0} + \frac{1}{2}\frac{P_2^2}{z_0}; \quad P_{Total} = \sqrt{2 z_0 I_{Total}}; \quad P_{Total}^{rms} = \sqrt{z_0 I_{Total}}; \quad (1.21)$$

Or simply $P_{Total} = \sqrt{P_1^2 + P_2^2}, P_{Total}^{rms} = \sqrt{P_{1,rms}^2 + P_{2,rms}^2} = 2.0\,\text{Pa}.$

**4. The Propagation of Sound in Time and Space: The Wave Equation.**

A wave equation describes the variation of an acoustic variable in time and space. The wave equations we will be using use the basic assumptions of linear acoustics (i.e. the sound-induced variations in p,T and ρ are small compared to P0,T0 and ρ0) and also assumes that air (and water) are inviscid (i.e. that sound propagation is primarily determined by the density and compressibility of the fluid and is not dependent on fluid viscosity).

*The one dimensional wave equation for sound pressure:*

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{\rho_0}{B_A}\frac{\partial^2 p(x,t)}{\partial t^2}, \text{ where } \frac{\rho_0}{B_A} = \frac{1}{c^2}. \quad (1.22)$$

The derivation of the one dimensional wave equation for sound in a fluid includes:

(i) An acoustic version of Newton's second law

$$\frac{\partial p(x,t)}{\partial x} = -\rho_0 \frac{\partial v_x(x,t)}{\partial t}$$

(ii) an acoustic version of the conservation of mass:

$$\rho_0 \frac{\partial v_x(x,t)}{\partial x} = -\frac{\partial \rho(x,t)}{\partial t}$$

(iii) a relationship between compressibility and sound-induced changes in density and pressure:

$$\rho(x,t) = p(x,t)\frac{\rho_0}{B_A}.$$

Those of you interested in the full derivation can look at the following pages.

**Topic 2: The Equations of Linear Acoustics; Derivation of the wave equation**

Advanced Reading: Beranek Chapter 2; Kinsler et al. Chapter 2
Another view: Fletcher, Chapter 6

*A. Define acoustic variables*

sound pressure: $p(x,y,z,t) = p_{Total}(x,y,z,t) - P_0$,

particle velocity: $\bar{v}(x,y,z,t) = v_x(x,y,z,t)\bar{i}_x + v_y(...)\bar{i}_y + v_z(...)\bar{i}_z$,

sound density: $\rho(x,y,z,t) = \rho_{Total}(x,y,z,t) - \rho_0$ and

sound temperature: $T(x,y,z,t) = T_{Total}(x,y,z,t) - T_0$

*B. The 'linear' assumption*

The primary assumption of linear acoustics is that all variations in acoustic scalar quantities are small relative to the static equilibrium quantities, i.e.

$$p(x,y,z,t) << P_0; \quad \rho(x,y,z,t) << \rho_0; \quad T(x,y,z,t) << T_0$$

*C. The 'inviscid' assumption*

Another common simplifying assumption is that the viscosity of air is so small that we can consider the medium inviscid. This assumption is generally valid except for tubes of very small cross-sectional area, when determining the '*Q*' of a resonant acoustic system (which depends strongly on damping) or considering the propagation of sound over some large distances.

*D. Derivation of the one-dimensional wave equation*

Assume a sound wave propagating in the *x* direction down a long rectangular duct of cross-sectional area *S* (Figure 1.2.1), with a 'wave-front' orthogonal to the long axis. Under these conditions, the sound pressure and particle velocity within any slice where *x* is constant are *invariant* and we can describe the system in terms of variations in a single dimension:

$$p(x,y,z,t) \Rightarrow p(x,t), \text{ and} \tag{1.2.1a}$$

$$\bar{v}(x,y,z,t) \Rightarrow v_x(x,t)\bar{i}_x + 0 \times \bar{i}_y + 0 \times \bar{i}_z \Rightarrow v_x(x,t). \tag{1.2.1b}$$

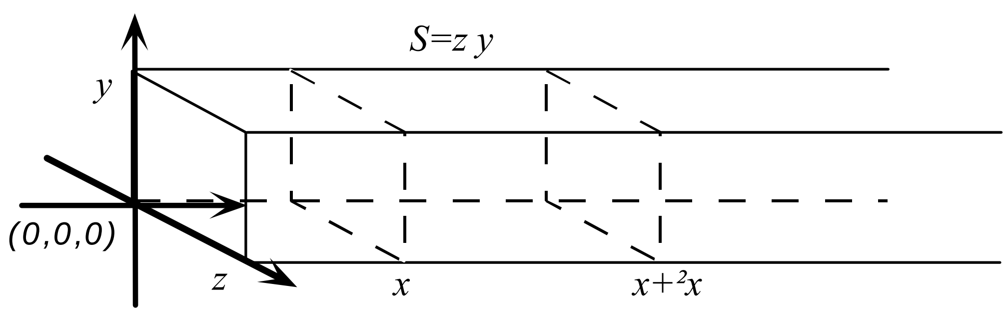



Figure 1.2.1: A long duct of height *y*, width *z* and undetermined length. Our derivation of the wave equation is based on a section of duct described by the interval *x* to *x*+Δ*x*.

In order to characterize the two unknowns $p(x,t)$ and $v_x(x,t)$, we need two constraining equations. We use Newton's second law as one constraint and the conservation of mass together with the elastic properties of the medium for the second constraint.

1. Constraint 1: Newton's second law:

Consider a small section of the duct of length $\Delta x$.. Newton's second law reads:

$$Force = m\frac{dv_x(t)}{dt} \quad . \tag{1.2.2}$$

The net force acting on the air in the small section of the duct is:

$$\begin{aligned} Force &= S \times (\text{Difference in pressure at } x \text{ and } \Delta x) \\ &= S(p(x,t) - p(x + \Lambda x, t)) \end{aligned} \quad . \tag{1.2.3}$$

The right side of equation (1.2.2) can be described in terms of the volume of the section of duct ($S\Delta x$) and the total density of air, leading to:

$$S(p(x,t) - p(x + \Delta x, t)) = S\Delta x \, \rho_{Total}(x,t)\frac{dv_x(x,t)}{dt} \quad . \tag{1.2.4a}$$

Substituting $\rho_{Total}(x,t)=\rho 0+\rho(x,t)$ into (1.2.4a) yields:

$$S(p(x,t) - p(x + \Delta x, t)) = S\Delta x \, (\rho_0 + \rho(x,t))\frac{dv_x(x,t)}{dt} \tag{1.2.4b}$$

Dividing each side of (1.2.4b) by the volume ($S\Delta x$), assuming $\rho(x,t) << \rho_0$ and taking the limit as $\Delta x$ goes to zero, yields:

$$\frac{\partial p(x,t)}{\partial x} = -\rho_0 \frac{dv_x(x,t)}{dt} \quad . \tag{1.2.5}$$

The derivative on the right of (1.2.5) can be rewritten in terms of partial derivatives as

$$\frac{dv_x(x,t)}{dt} = v_x(x,t)\frac{\partial v_x(x,t)}{\partial x} + \frac{\partial v_x(x,t)}{\partial t} \quad , \text{ and}$$

since we assume $v_x(x,t)$ is small:

$$\frac{dv_x(x,t)}{dt} \approx \frac{\partial v_x(x,t)}{\partial t} \quad . \tag{1.2.6}$$

The final result relates pressure and velocity via two first order differential equations, one in space and the other in time:

$$\frac{\partial p(x,t)}{\partial x} = -\rho_0 \frac{\partial v_x(x,t)}{\partial t} \, . \tag{1.2.7}$$

**The acoustic version of F=ma!!**

2. Constraint Two:

*a. The conservation of mass.*

In a sound field we can also define a flux $\bar{J}$ (the mass flowing per unit area per unit time) as the product of the total density and the particle velocity

$$\bar{J} = \rho_{Total}\bar{v}. \qquad (1.2.8)$$

(The units of flux are kg/m$^2$/s).

In the one dimensional system of Figure 1.2.1, we need only worry about the *x* component of the vectors $\bar{J}$ and $\bar{v}$; and we can rewrite (1.2.8) as:

$$J_x(x,t) = \rho_{Total}(x,t)v_x(x,t) \quad . \qquad (1.2.9)$$

According to the conservation of mass, the net flux into a volume (SΔx) during time Δt must equal the change in mass defined by the product of the volume and the change in total density during time Δt:

$$\left(J_x(x,t) - J_x(x+\Delta x,t)\right)\Delta t\, S = \left(\rho_T(x,t+\Delta t) - \rho_T(x,t)\right)\Delta x\, S. \qquad (1.2.10)$$

Dividing both sides of (1.2.10) by Δx Δt S, and taking the limit as both Δx and Δt go to zero yields;

$$\frac{\partial J_x(x,t)}{\partial x} = -\frac{\partial \rho_{Total}(x,t)}{\partial t}. \qquad (1.2.11)$$

Substituting (1.2.9) into (1.2.11) and noting that $\frac{\partial \rho_{Total}}{\partial t} = \frac{\partial \rho}{\partial t}$;gives us our final description of the conservation of mass or continuity equation for a one-dimensional linear acoustic system:

$$\rho_0 \frac{\partial v_x(x,t)}{\partial x} = -\frac{\partial \rho(x,t)}{\partial t} \quad . \qquad (1.2.12)$$

**Given a 'box' of medium, the change in density as a function of time is proportional to the net velocity of particles entering and leaving the box.**

b. The elastic properties of the medium

In general the density of a gas (like air) depends on pressure. For example, the Ideal Gas Law relates the total pressure pT and volume Vol of the gas to some physical constants, n the number of moles of the gas, R the universal gas constant and TTotal the absolute temperature:

$$p_{Total}(x,t)\ Vol = nRT_{Total}(x,t)\ . \tag{1.2.13}$$

The gas law can be rewritten in terms of density ρ:

$$p_{Total} = \rho_{Total} \frac{RT_{Total}}{M}, \tag{1.2.14}$$

where $M$ is the molecular weight of the gas.

If $T_{Total}$ were to remain constant (isothermal conditions) then the relationship between $p_{Total}$ and $\rho_{Total}$ is especially simple, such that:

$$\frac{p_{Total}}{p_0} = \frac{\rho_{Total}}{\rho_0}\ . \tag{1.2.15}$$

However, the rapid changes of pressure associated with sound at most audible frequencies does not permit heat exchange either within the sound field or between the sound field and the environment. Therefore the relation between $p_{Total}$ and $\rho_{Total}$ must be defined for circumstances of no heat flow (adiabatic conditions). For an ideal gas under adiabatic conditions:

$$\frac{p_{Total}}{p_0} = \left(\frac{\rho_{Total}}{\rho_0}\right)^{\gamma}, \tag{1.2.16}$$

where γ is the ratio of specific heats and equals 1.41 for an ideal diatomic gas. Equation 1.2.16 can be rewritten to include acoustic variables:

$$\frac{p_0 + p}{p_0} = \left(\frac{\rho_0 + \rho}{\rho_0}\right)^{\gamma} \quad \text{or} \quad p_0 + p = p_0\left(\frac{\rho_0 + \rho}{\rho_0}\right)^{\gamma}\ . \tag{1.2.17}$$

A Taylor series expansion of the right side of (1.2.17) yields:

$$p_0 + p = p_0 + \rho \frac{\partial p_{Total}}{\partial \rho_{Total}} + \frac{\rho^2}{2} \frac{\partial^2 p_{Total}}{\partial \rho^2_{Total}} + \ldots \tag{1.2.18}$$

Since ρ is small the higher terms can be ignored, leaving

$$p(x,t) = \rho(x,t) \frac{\partial p_{Total}}{\partial \rho_{Total}} = B_A \frac{\rho(x,t)}{\rho_0} \quad \text{or} \quad \rho(x,t) = p(x,t) \frac{\rho_0}{B_A} \tag{1.2.19}$$

where $B_A = \rho_0 \frac{\partial p_{Total}}{\partial \rho_{Total}}$. For an ideal gas, $B_A = \gamma p_0$. For a diatomic gas at 1 atmosphere of pressure $B_A$=1.4x$10^5$ Pa.

**Directly from the gas law: P = n/V RT, or n/V = P/(RT).**

3. Synthesis:

Newtons 2nd Law: $$\frac{\partial p(x,t)}{\partial x} = -\rho_0 \frac{\partial v_x(x,t)}{\partial t} \quad . \qquad (1.2.7)$$

Conservation of Mass: $$\rho_0 \frac{\partial v_x(x,t)}{\partial x} = -\frac{\partial \rho(x,t)}{\partial t} \quad . \qquad (1.2.12)$$

Elasticity Relationship: $$\rho(x,t) = p(x,t)\frac{\rho_0}{B_A} \quad . \qquad (1.2.19)$$

Substituting 19 into 12 yields:

$$\frac{\partial v_x(x,t)}{\partial x} = -\frac{1}{B_A}\frac{\partial p(x,t)}{\partial t} \quad . \qquad (1.2.20)$$

Equations (1.2.7) and (1.2.20) describe the partial derivatives of pressure and velocity with respect to *x* in terms of two equations, two properties of the media ($\rho_0$ and B) and two unknowns (the partial derivatives with respect to time). Taking the partial of both sides of (1.2.7) with respect to *x* and substituting (1.2.20) into the result yields a one-dimensional "wave equation" in *x* and *t,*

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{\rho_0}{B_A}\frac{\partial^2 p(x,t)}{\partial t^2}, \text{ where } \frac{\rho_0}{B_A} = \frac{1}{c^2}. \qquad (1.2.21)$$

One possible solution for (1.2.21) is in terms of two wave functions traveling in opposite directions:

$$p(x,t) = f^+(t - x/c) + f^-(t + x/c) \quad , \qquad (1.2.22a)$$

and $$v_x(x,t) = \frac{1}{z_0}\left[f^+(t - x/c) - f^-(t + x/c)\right] \quad , \qquad (1.2.22b)$$

Things to note about 1.2.22

1). $z_0 = \sqrt{B_A \rho_0} = \rho_0 c$ ,
2). The wave functions $f^+$ & $f^-$ are functions in time and space but the units to function argument is seconds.
3.) In the forward *traveling wave* $f^+$, for any *t* the argument *(t-x/c)* is smaller for larger *x*s.
4.) In the backward *traveling wave* $f^-$ for any *t* the argument *(t+x/c)* is larger for larger *x*s.
5.) While the scalar pressures produced by the two oppositely traveling waves add, the directional velocities are of opposite sign and subtract.
6.) While the pressure and velocity components of each traveling wave are related by $z_0$, the sum terms $p(t,x)/v_x(t,x)$ may not be.