1. (35 points)

A surface current of infinite extent in the $y$ and $z$ directions, $\vec{K}(x \ 0, \ t) = K_0 \cos \omega t \vec{I}_z$, is located in the $x = 0$ plane between a region with infinite magnetic permeability, $\mu \to \infty$ for $x < 0$, and a superconductor in the region $0 < x < d$ described by plasma radian frequency $\omega_p$, dielectric permittivity $\varepsilon_0$, and magnetic permeability $\mu_0$. The region for $x > d$ is a perfect conductor with $\sigma \to \infty$.

In the superconducting region, $0 < x < d$, the volume free current density $\vec{J}$ is related to the electric field $\vec{E}$ as

$$\frac{\partial \vec{J}}{\partial t} = \omega_p^2 \varepsilon_0 \vec{E}$$

and the magnetic field $\vec{H}$ obeys the equation

$$\nabla^2 \vec{H} - \omega_p^2 \varepsilon_0 \mu_0 \vec{H} = 0$$

Because the system is uniform and of infinite extent in the $y$ and $z$ directions, all fields only depend on the $x$ coordinate.

(a) What is the direction of $\vec{H}$ for $0 < x < d$ and what is $\vec{H}$ for $x < 0$ and $x > d$?

(b) What boundary conditions must $\vec{H}$ satisfy at $x = 0$ and $x = d$?

(c) What are the directions of $\vec{J}$ and $\vec{E}$ in the region $0 < x < d$? What boundary condition must $\vec{E}$ satisfy at $x = d$?

(d) What is the general form of solution of $\vec{H}$, $\vec{J}$, and $\vec{E}$ for $0 < x < d$?

(e) Apply the boundary conditions of (b) and (c) and solve for $\vec{H}$, $\vec{J}$, and $\vec{E}$.

(f) What is the surface current $\vec{K}(x \ d, t)$ on the $x = d$ plane?

(g) What is the force per unit area on the $x = d$ plane?
A permanently magnetized medium of infinite extent in all directions has magnetization:
\[ M = M_0 \bar{I}_z = M_0 \left[ \bar{I}_z \cos \theta - I_0 \sin \theta \right] \]

The medium has zero conductivity so that the free volume current density, \( \bar{J} \), is zero everywhere. The medium has a spherical hole of radius \( R \) filled with free space with magnetic permeability \( \mu_0 \). The magnetic field \( \bar{H} \) as radius \( r \) from the center of the spherical hole goes to infinity is zero, \( \bar{H}(r \to \infty) = 0 \). The field solutions are axisymmetric so that there is no variation with the azimuthal angle \( \phi \).

(a) Because \( \bar{J} \equiv 0 \), the magnetic field \( \bar{H} \) has zero curl. This allows the definition of a magnetic scalar potential \( \chi(r, \theta) \) where \( \bar{H} = -\nabla \chi \). What are the governing equations for \( \chi \) in the regions \( r < R \) and \( r > R \)?

(b) What boundary conditions must be satisfied at \( r = R \)?

(c) Solve for the magnetic scalar potential \( \chi(r, \theta) \) and magnetic field \( \bar{H}(r, \theta) \) in regions \( r < R \) and \( r > R \).

(d) For \( r > R \), what is the effective magnetic dipole moment of the spherical hole? Hint: The magnetic scalar potential for a point magnetic dipole with moment \( \bar{m} \) directed in the \( z \) direction is \( \chi(r, \theta) \equiv \frac{\bar{m} \cos \theta}{4\pi r^2} \).
The figure illustrates a magnetic yoke with infinite permeability; and an air gap with magnetic permeability $\mu_0$, length $g$, and cross-sectional area $Dw$. An incompressible block with length $l \gg D$, magnetic permeability $\mu$ and cross-sectional area $w_g$ can move in the $x$ direction as shown. Both the magnetic block and the yoke can be assumed to have negligible electrical conductivity. There is an $N$ turn winding around the magnetic yoke driven by a DC current source $I_0$. Assume throughout that the position $x$ of the moveable block is bounded as $0 \leq x \leq D$.

(a) Find the flux linkage ($\lambda$) – current ($i$) relationship for the winding in terms of $x$, $\mu$, $\mu_0$, $N$ and the dimensions of the magnetic circuit.

(b) Find the force in the $x$ direction on the incompressible block due to the current source $I_0$.

(c) If the magnetic block is forced to move with a displacement given by $x = x_0 + \Delta x \cos(\omega t)$ find the voltage developed across the current source.

(d) The current source and the block position are controlled to traverse the $i$ – $x$ plane in a cycle as shown. Find the work done by the current source for each cycle.