Lecture 5: Classical Model of a Superconductor

Outline

1. First and Second London Equations
2. Examples
   - Superconducting Slab
   - Bulk Sphere
3. Non-simply connected superconductors
   - Hollow cylinder
   - Superconducting circuits
     - DC flux transformer
     - Superconducting memory loop
     - Magnetic monopole detector
4. Two Fluid Model

Superconductor: Classical Model

\[
E = \frac{\partial}{\partial t} (\Lambda J) \quad \text{first London Equation}
\]

\[
\nabla \times (\Lambda J) = -B \quad \text{second London Equation}
\]

\[
\Lambda = \frac{m^*}{n^*(q^*)^2} \quad \lambda = \sqrt{\frac{\Lambda}{\mu_0}} \quad \text{penetration depth}
\]

When combined with Maxwell’s equation in the MQS limit

\[
\left(\frac{1}{\lambda^2} - \nabla^2\right) \mathbf{H} = 0
\]
Superconducting Infinite Slab

Let \( H(r, t) = \Re \left\{ \bar{H}(y) e^{i\omega t} \right\} i_z \)

Therefore,
\[
\left( \frac{1}{\lambda^2} - \frac{d^2}{dy^2} \right) \bar{H}(y) = \mathcal{C}
\]

and
\[
\bar{H}(y) = C \cosh(y/\lambda)
\]

\( H_{\text{app}} = \Re \left\{ \bar{H}_0 e^{i\omega t} \right\} i_z \)

Boundary Conditions demand
\[
\left( \frac{1}{\lambda^2} - \nabla^2 \right) H = 0 \quad H_z(a) = H_z(-a) = C \cosh(a/\lambda) = \bar{H}_c
\]

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Fields and Currents for \(|y| < a\)

\[
H = \Re \left\{ \bar{H}_0 \frac{\cosh(y/a)}{\cosh(a/\lambda)} e^{i\omega t} \right\} i_z \quad J = \Re \left\{ \bar{H}_0 \frac{\sinh(y/\lambda)}{\lambda \cosh(a/\lambda)} e^{i\omega t} \right\} i_x
\]

Thin film limit
\[ a \ll \lambda \]

Bulk limit
\[ a \gg \lambda \]
Superconducting Sphere: Bulk Approximation $R \gg \lambda$

\[ H(r \leq R) = 0 \]
\[ H(r \geq R) = \text{Re} \left\{ \bar{H}_0 \left( 1 - \left( \frac{R}{r} \right)^3 \right) \cos \theta e^{j\omega t} \right\}_r - \text{Re} \left\{ \bar{H}_0 \left( 1 + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right) \sin \theta e^{j\omega t} \right\}_\theta \]
\[ K(r = R) = -\text{Re} \left\{ \frac{3}{2} \bar{H}_0 \sin \theta e^{j\omega t} \right\}_\phi \]

Current along a cylinder: bulk superconductor

The fields from Ampere’s law
\[ \oint_C H \cdot dl = \int_S J \cdot ds \]

Inside: \[ H(2\pi r) = 0 \]

Outside: \[ H(2\pi r) = I \]

\[ I = \text{Re} \left\{ \bar{I}_0 e^{j\omega t} \right\}_z \]

\[ J(r \leq R) = \frac{1}{\pi R^2} \]

Therefore, \[ K(r = R) = \frac{I}{\pi R} \]

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Field along a cylinder: bulk superconductor

\[
\begin{align*}
H_{\text{out}} &= \text{Re}\left\{ H_0 e^{j\omega t} \right\} i_z \\
H_{\text{app}} &= \text{Re}\left\{ H_0 e^{j\omega t} \right\} i_z \\
H_{\text{in}} &= 0 \\
K(r=R) &= -H_0 i_\phi
\end{align*}
\]

Field along a hollow cylinder

Solution 1

\[
H_{\text{app}} = \text{Re}\left\{ H_0 e^{j\omega t} \right\} i_z
\]

or

Solution 2

\[
H_{\text{app}} = \text{Re}\left\{ H_0 e^{j\omega t} \right\} i_z
\]
Multiply Connected Superconductor

Maxwell \[ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \]

First London \[ \oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int_C \mathbf{A} \cdot d\mathbf{l} \]

Therefore \[ \frac{d}{dt} \left[ \Phi + \int_C \mathbf{A} \cdot d\mathbf{s} \right] = 0 \]

and \[ \Phi + \int_C \mathbf{A} \cdot d\mathbf{s} = \Phi_C = \text{constant} \]

For a contour within the bulk where \( \mathbf{J} = 0 \), flux remains constant

Field along a hollow cylinder

Zero Field
Initially Solution

Finite Field
Initially Solution

\[ \mathbf{H}_{\text{app}} = \Re \left\{ \hat{H}_0 e^{j\omega t} \right\} \mathbf{i}_z \]
A generalization to any closed superconducting circuit is that the total flux linkage in a circuit remains constant.

Then if a circuit has $N$ elements that can contain flux,

$$\lambda_{\Phi_1} + \lambda_{\Phi_2} + \ldots = \text{constant}$$

Sources of Flux linkage

$$\lambda_{\Phi_a} = L_a i_a + M_{ab} i_b + \ldots + \lambda_{\Phi_{\text{ext}}}$$

- Self-inductance
- Mutual inductance
- External flux
**DC Flux Transformer**

If the $B$ field is measured of the transported flux

$$\beta = \frac{|B_2|}{|B_{app}|} = \frac{N_1 A_1}{N_2 A_2} \frac{L_2}{L_1 + L_2}$$

B can be amplified

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**Superconducting Memory**

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Magnetic Monopole Detector

Maxwell’s Equations with Monopole density $\rho_m$

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{J}_r \\
\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_e
\end{align*}
\]

$\nabla \cdot \mathbf{D} = \rho_e$

$\nabla \cdot \mathbf{B} = \rho_m$

The signs insure electric and magnetic charge conservation.

\[
\begin{align*}
\nabla \cdot \mathbf{J}_e + \frac{\partial}{\partial t} \rho_e &= 0 \\
\nabla \cdot \mathbf{J}_m + \frac{\partial}{\partial t} \rho_m &= 0
\end{align*}
\]
Magnetic Monopole Detector

Take the line integral

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{J}_m \]

\[ - \oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S \mathbf{B} \cdot d\mathbf{s} + \int_S \mathbf{J}_m \cdot d\mathbf{s} \]

\[ 0 = \frac{d \Phi}{dt} + I_m \]

Total of Flux and magnetic charge is conserved.

Inductance measurement

From the measurement of the inductance, the penetration depth can determined.

For a normal metal

\[ \Phi = \frac{N}{I} N \pi R^2 \]

And

\[ L = \frac{N^2}{L} \pi R^2 \]

For a superconductor,

\[ \Phi = \frac{N}{I} N 2\pi R \lambda \]

\[ L = \frac{N^2}{I} 2\pi R \lambda \]
The penetration depth $\lambda$ is temperature dependent!

$\lambda(T) = \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m^*}{n^*(q^*)^2 \mu_0}} = \frac{\lambda_0}{\sqrt{1 - (T/T_C)^4}}$ for $T \leq T_C$.

A good guess to let $n^*$ depend on temperature for $T < T_C$

$$n^*(T) = \frac{1}{2} n_{tot} \left( 1 - \left( \frac{T}{T_C} \right)^4 \right)$$

$$n_{tot} = n(T) + 2n^*(T)$$

$$n(T) = n_{tot} \left( \frac{T}{T_C} \right)^4$$
Two Fluid Model for $\omega \tau_{tr} << 1$, $T < T_c$

$$\mathbf{J}_{\text{tot}} = \mathbf{J}_s(T) + \mathbf{J}_n(T)$$

$$\mathbf{E} = \frac{\partial}{\partial t} (\Lambda(T) \mathbf{J}_s) \quad \mathbf{E} = \frac{1}{\bar{\sigma}_o(T)} \mathbf{J}_r$$

$$\Lambda(T) = \frac{m}{n_{\text{tote}}^2 e^2} \left( \frac{1}{1 - (T/T_c)^4} \right) \quad \bar{\sigma}_o(T) = \frac{n_{\text{tote}} e^2 \tau_{tr}}{m} \left( \frac{T}{T_c} \right)^4$$

$$\Lambda(T) \quad \frac{1}{\bar{\sigma}_o(T)}$$

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s = \left( \bar{\sigma}_o(T) + \frac{1}{j \omega \mu_o (\lambda(T))^2} \right) \mathbf{E}$$

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Two Fluid Model

Constitutive relations for two fluid model

$$\mathbf{E} = \frac{\partial}{\partial t} (\Lambda(T) \mathbf{J}_s) \quad \mathbf{E} = \frac{1}{\bar{\sigma}_o(T)} \mathbf{J}_r$$

$$\nabla \times (\Lambda(T) \mathbf{J}_s) = -\mathbf{B}$$

Maxwell

$$\nabla \times \mathbf{H} \approx \mathbf{J} = \mathbf{J}_n + \mathbf{J}_s \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}$$

Gives

$$\left( 1 - \lambda^2 \nabla^2 + \mu_o \bar{\sigma}_o \lambda^2 \frac{\partial}{\partial t} \right) \mathbf{B}$$
Complex wavenumber

For a sinusoidal drive,

\[
\left( 1 - \lambda^2(T) \nabla^2 + j 2 \left( \frac{\lambda(T)}{\delta(T)} \right)^2 \right) \vec{B} = 0
\]

For a slab in a uniform field

\[
\vec{B} = \mu_0 \vec{H}_0 \frac{\cosh ky}{\cosh k\alpha} \hat{z}.
\]

\[
(k(T))^2 = \frac{1}{(\lambda(T))^2} + j \frac{2}{(\delta(T))^2}
\]

The smaller length determines the length scale