The Confluence of the \( \lambda \)-calculus

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Lecture 23

Confluence aka Church-Rosser Property

A reduction system \( R \) is said to be confluent (CR), if \( t \rightarrow t_1 \) and \( t \rightarrow t_2 \), then there exists a \( t_3 \) such that \( t_1 \rightarrow t_3 \) and \( t_2 \rightarrow t_3 \).

Fact: In a confluent system, if a term has a normal form then it is unique.

Theorem: The \( \lambda \)-calculus is confluent.

Theorem: An orthogonal TRS is confluent.
The Diamond Property

A reduction system $R$ is said to have the diamond property, if $t \rightarrow t_1$ and $t \rightarrow t_2$ then there exits a $t_3$ such that $t_1 \rightarrow t_3$ and $t_2 \rightarrow t_3$.

![Diagram](http://www.csg.lcs.mit.edu/6.827)

**Theorem:** If $R$ has the diamond property then $R$ is confluent.

**Fact:** The $\lambda$-calculus does not have the diamond property.

Weak Confluence

A reduction system $R$ is said to be weakly confluent (WCR), if $t \rightarrow t_1$ and $t \rightarrow t_2$ then there exits a $t_3$ such that $t_1 \rightarrow t_3$ and $t_2 \rightarrow t_3$.

![Diagram](http://www.csg.lcs.mit.edu/6.827)

*In a WCR system one step divergence can be contained!*

**Theorem:** If $R$ is CR then $R$ is also WCR.

**Theorem:** If $R$ is WCR then $R$ is also WCR.
WCR does not imply CR

Example:

\[ F(x) \rightarrow G(x) \]
\[ F(x) \rightarrow 1 \]
\[ G(x) \rightarrow F(x) \]
\[ G(x) \rightarrow 0 \]

Why WCR does not imply CR

Suppose R is WCR
Completing this diagram looks like proving the CR theorem again!

The diagram may not complete!
There will be no problem if all the reduction paths were finite
Strongly Normalizing Systems

Let \((\Sigma, R)\) be a TRS and \(t\) be a term

\(t\) is in normal form if it cannot be reduced any further.

Term \(t\) is strongly normalizing (SN) if every reduction sequence starting from \(t\) terminates eventually.

R is strongly normalizing (SN) if for all terms every reduction sequence terminates eventually.

R is weakly normalizing (WN) if for all terms there is some reduction sequence that terminates.

Neumann's Lemma

If a reduction system \(R\) is SN and WCR then \(R\) is CR.

How does it help us when an \(R\) is not SN?

Only “old” redexes need to be performed to close the diagram

\(\Rightarrow\) define a new reduction system for doing just the “old” redexes.

Is such a system SN?
Underlining and Development

Underline some redexes in a term.

Development is a reduction of the term such that only underlined redexes are done.

Complete Development is a reduction sequence such that all the underlined redexes have been performed.

\[
\begin{align*}
& (S \ K \times (K \ y \ z)) \\
\rightarrow & (S \ K \times y) & \rightarrow K \ (K \ y \ z) \ (x \ (K \ y \ z)) \\
\rightarrow & K \ y \ (x \ y) & \rightarrow K \ y \ (x \ (K \ y \ z)) \\
\rightarrow & K \ y \ (x \ y)
\end{align*}
\]

By underlining redexes we can distinguish between old and newly created redexes in a reduction sequence.

The Underlined \(\lambda\)-calculus

\[
E = x \mid \lambda x.E \mid E \ E \mid (\lambda x.E) \ E
\]

Reduction rules:

\[
\begin{align*}
\beta: & (\lambda x.M) \ A \rightarrow M[A/x] \quad \text{the } \lambda\text{-calculus} \\
\beta': & (\lambda x.M) \ A \rightarrow M[A/x] \quad \text{the } \lambda\text{-calculus} \\
\end{align*}
\]

Erasure:

? Er? \(\lambda\)-term \(\rightarrow\) \(\lambda\)-term

Facts:

\[
\begin{align*}
M & \xrightarrow{\beta'} N \\
M & \xrightarrow{\beta} N \\
M & \xrightarrow{Er} N \\
M & \xrightarrow{Er} N
\end{align*}
\]
**Complete Development An Example**

\[ M = (\lambda x.x) (I (I a)) \]  
where \( I = (\lambda x.x) \)

Underline some redexes

\[ M = (\lambda x.x) (I (I a)) \]

\[ \rightarrow (I (I a)) (I (I a)) \]
\[ \rightarrow (I a) (I (I a)) \]
\[ \rightarrow (I a) (I a) \]

\[ \rightarrow (\lambda x.x) (I a) \]
\[ \rightarrow (I a) (I a) \]

**Underlined Reduction Systems are SN**

*Theorem*: For every reduction system \( R \), \( R \) is strongly normalizing.

Proof strategy:
Assign a *weight* to each term \( M \) such that the weight decreases after each reduction.

\[ M \rightarrow N \Rightarrow |N| < |M| \]
where \( |M| \) represents the weight of \( M \).

Thus, if
\[ M \rightarrow M_1 \rightarrow M_2 \rightarrow \ldots \]
\[ \Rightarrow |M| > |M_1| > |M_2| > \ldots \]
\[ \Rightarrow \text{since for all } M, |M| > 0, \text{ the reduction terminates!} \]

*Decreasing weight property*
Assigning Weights (The $\lambda$–calculus)

Associate a positive integer to each variable occurrence in $M$

$| M |$: sum of the weights occurring in $M$

$| x^w | = w$

$| \lambda x.M | = | M |$

$| \lambda x.M | = | M |$

$| M N | = | M | + | N |$

Weights, like underlined $\lambda$, are carried through the reduction unchanged.

Decreasing Weight Property (dwp)

$M$ has decreasing weight property if for every $\beta$ - redex $((\lambda x.P) Q)$ in $M$, $|x| > |Q|$ for each free occurrence of $x$ in $P$

Examples

$M_1 = (\lambda x. x^6 x^7) (\lambda y. y^2 y^3)$

$M_2 = (\lambda x. x^4 x^7) (\lambda y. y^2 y^3)$
**Initial Weight Assignment**

*Lemma:* There exits an initial weight assignment for each $M$ such that $M$ has dwp.

*Proof:*
1. Assign the weight $2^m$ to the $m^{th}$ variable occurrence from the right

   \[ M = \cdots x \cdots \implies |x| = 2^m \]

2. $M$ has the dwp since

   \[ 2^n > 2^{n-1} + 2^{n-2} + \ldots + 1 \]

Example:

\[(x \ y \ ((\lambda z.z) \ (x \ x)))\]

**Reduction Decreases the Weight of a term with dwp**

*Lemma:* If $M$ has dwp and $M \rightarrow N$ then $|N| < |M|$

*Proof:*
Suppose $(\lambda x. P) \ Q$ is the redex that is reduced when $M \rightarrow N$.

Cases
(i) $x$ is not in FV($P$):

(ii) $x$ is in FV($P$):

http://www.csg.lcs.mit.edu/6.827
Lemma: If M --> N and M has dwp then so does N.

Proof: Suppose M --> N by doing the redex \( R_0 \equiv (\lambda x. P_0) Q_0 \). Examine the effect of \( R_0 \)-reduction on some other redex \( R_1 \equiv (\lambda y. P_1) Q_1 \) in M.

Cases on relative position of \( R_0 \) and \( R_1 \)

1. \( R_0 \) and \( R_1 \) are disjoint
2. \( R_1 \) is inside \( R_0 \) (effect on subterms)
3. \( R_0 \) is inside \( R_1 \) (effect on the context)

Suppose M --> N by doing the redex \( R_0 \equiv (\lambda x. P_0) Q_0 \).
Examine the effect of \( R_0 \)-reduction on \( R_1 \equiv (\lambda y. P_1) Q_1 \).

Case 2. \( R_1 \) is inside \( R_0 \) (effect on subterms)

2.1 \( R_1 \) is inside the rator, \( \lambda x. P_0 \)
\[ R_0 \equiv (\lambda x. \cdots ((\lambda y. P_1) Q_1) \cdots) Q_0 \]

2.2 \( R_1 \) is inside the rand, \( Q_0 \)
\[ R_0 \equiv (\lambda x. P_0) (\cdots R_1 \cdots) \]
Suppose $M \rightarrow N$ by doing the redex $R_0 \equiv (\lambda x.P_0)Q_0$. Examine the effect of $R_0$-reduction on $R_1 \equiv (\lambda y.P_1)Q_1$.

Case 3. $R_0$ is inside $R_1$ (effect on the context)

3.1 $R_0$ is inside the rator of $R_1$

$$R_1 \equiv (\lambda y.\ldots((\lambda x.P_0)Q_0)\ldots)Q_1$$

3.2 $R_0$ is inside the rand of $R_1$

$$R_1 \equiv (\lambda y.P_1)(\ldots((\lambda x.P_0)Q_0)\ldots)$$

Proof Strategy for CR

Define a new type of reduction called complete developments (CD) using the underlined $\lambda$-calculus.

Prove the diamond property for CD reductions, i.e., show that CD is SN and CD is WCR.

The proof of confluence for the $\lambda$-calculus follows:

Each reduction can be viewed as a CD

Since CD reductions have the diamond property
\textbf{\(\lambda\)-calculus is WCR}

Suppose \(M \rightarrow M_1\) by doing redex \(R_1\) and \(M \rightarrow M_2\) by doing redex \(R_2\).

We want to show that there exists an \(M_3\) such that \(M_1 \rightarrow\rightarrow M_3\) and \(M_2 \rightarrow\rightarrow M_3\).

\textit{Cases} on relative position of \(R_1\) and \(R_2\) in \(M\).

1. \(R_1\) and \(R_2\) are \textit{disjoint}

2. Without loss of generality assume \(R_1\) is inside \(R_2\)
   - 2.1 \(R_1\) is in the rator of \(R_2\) from the substitution lemma
   - 2.2 \(R_1\) is in the rand of \(R_2\)

\textbf{Substitution Lemma}

If \(x\) is not equal to \(y\) and \(x\) is not in \(\text{FV}(L)\) then
\[
M \ [N/x] \ [L/y] = M \ [L/y] \ [N[L/y]/x]
\]

\((\lambda y. (\lambda x. M) N) L\)
Finite Development Theorem

Suppose $M$ is a $\lambda$-term and $F$ is a set of redexes in $M$, then

1. All developments of $M$ related to $F$ are finite

2. All complete developments of $M$ related to $F$ end with the same term.

*The proof follows from the fact that the $\lambda$-calculus is SN and WCR*

CD Reduction has the Diamond Property

$M \xrightarrow{F_1} M_1$

$M \xrightarrow{F_2} M_3$

$M_3$ is a CD of $M$ with respect to $F_1 \cup F_2$

$M \xrightarrow{F_2} M_2$

$M_2 \xrightarrow{F_1} M_3$
Orthogonal TRS

- Confluence of orthogonal TRS’s can be shown in the same way.

Orthogonal TRSs

A TRS is *Orthogonal* if it is:

1. *Left Linear*: has no multiple occurrences of a variable on the LHS of any rule, and

2. *Non Interfering*: patterns of rewrite rules are pairwise non-interfering

*Theorem*: An Orthogonal TRS is Confluent.
Orthogonal TRSs are CR

Proof outline:
1. $R$ is orthogonal $\Rightarrow R$ is orthogonal.
2. $R$ is orthogonal $\Rightarrow R$ is WCR $\Rightarrow \bar{R}$ is WCR.
3. $R$ is SN
4. From 2. and 3. $R$ is CR (Neumann's Lemma)
5. Transitive Closure of $R = \text{Transitive closure of} \bar{R} \Rightarrow R$ is CR.

If $R$ is orthogonal then $R$ is WCR

Case 1: $\alpha$ and $\beta$ are disjoint $\alpha$ and $\beta$ commute (trivially)

Case 2: $\alpha$ is a subexpression of $\beta$ $\Rightarrow \beta$ cannot be a subexpression of $\alpha$?

Case 2a: $\alpha$ is reduced before $\beta$
Since $R$ is orthogonal, reducing $\alpha$?
cannot affect $\beta$

Case 2b: $\beta$ is reduced before $\alpha$
$\beta$ can destroy or duplicate $\alpha$?

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