

# Quantum Computing with Noninteracting Particles

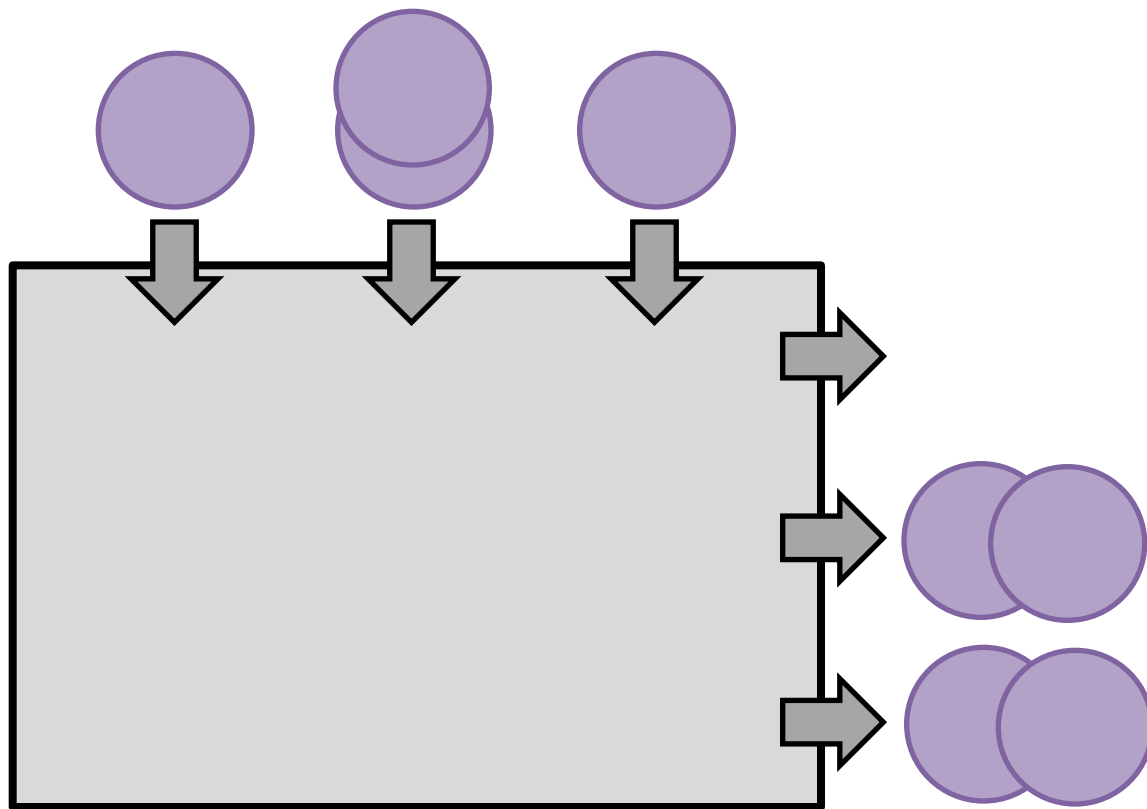
*Alex Arkhipov*

# Noninteracting Particle Model

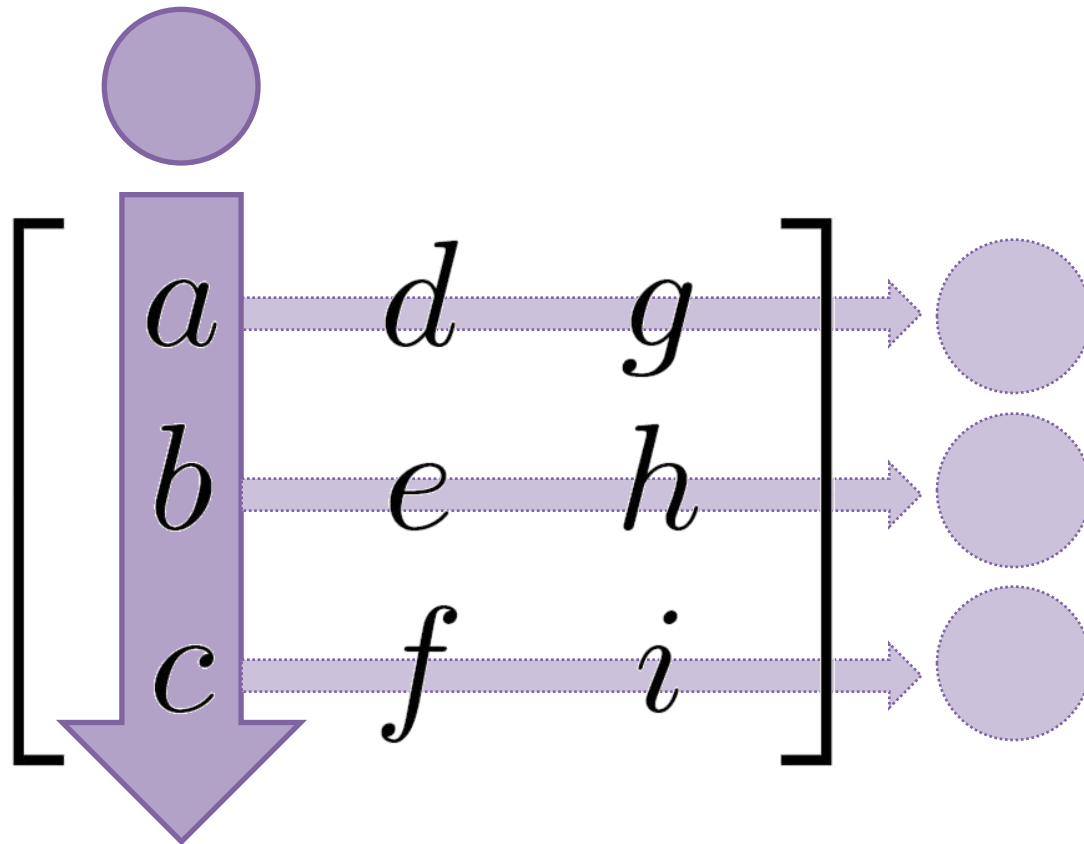
- Weak model of QC
  - Probably not universal
  - Restricted kind of entanglement
  - Not qubit-based
- Why do we care?
  - Gains with less quantum
  - Easier to build
  - Mathematically pretty

# Classical Analogue

# Balls and Slots



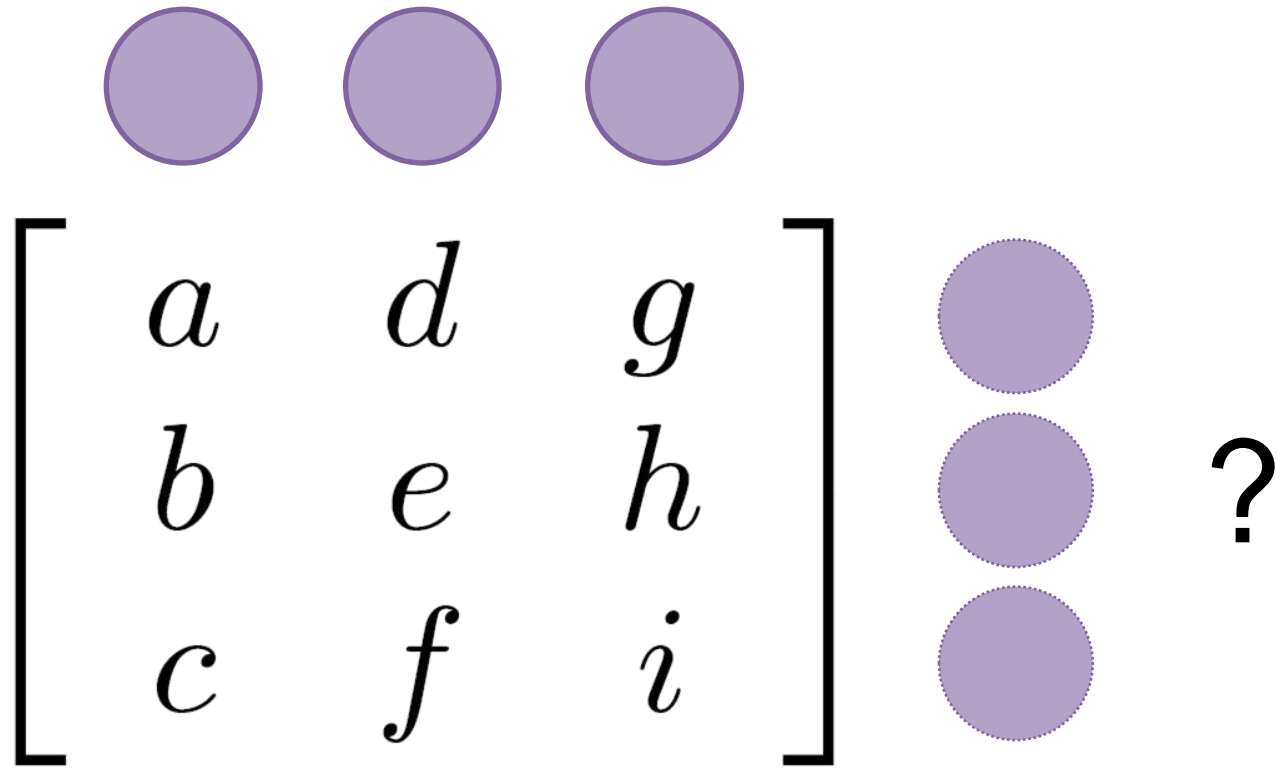
# Transition Matrix



$$a, b, c \geq 0$$

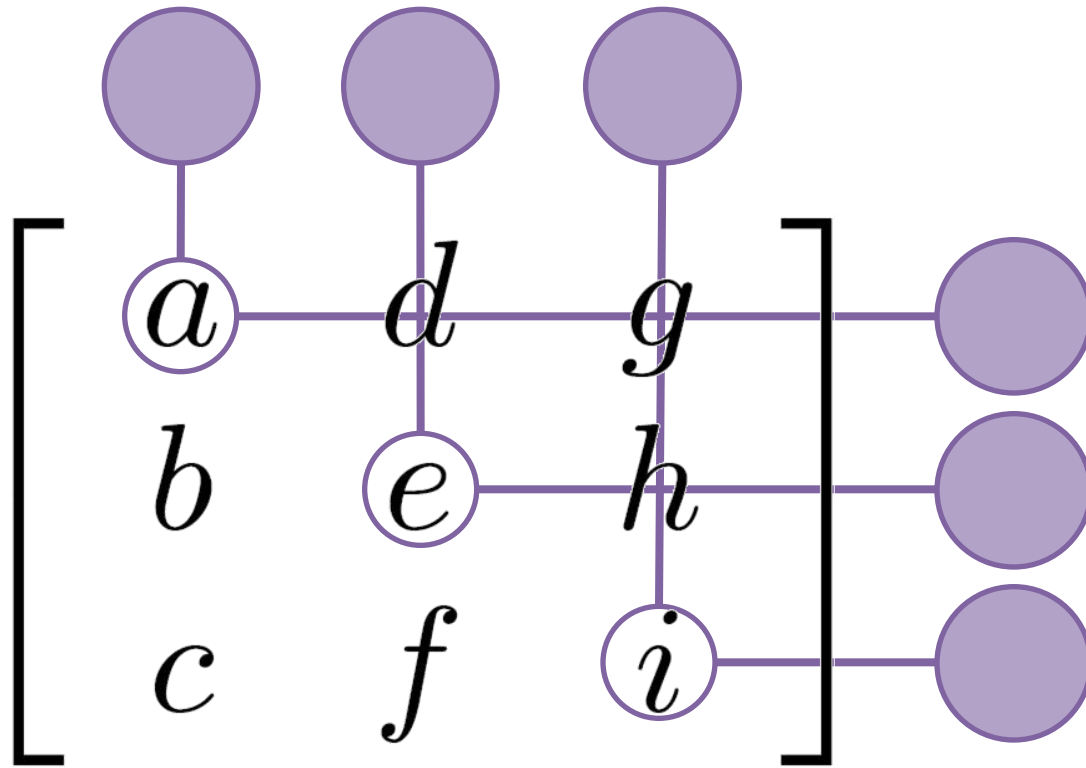
$$a + b + c = 1$$

# A Transition Probability



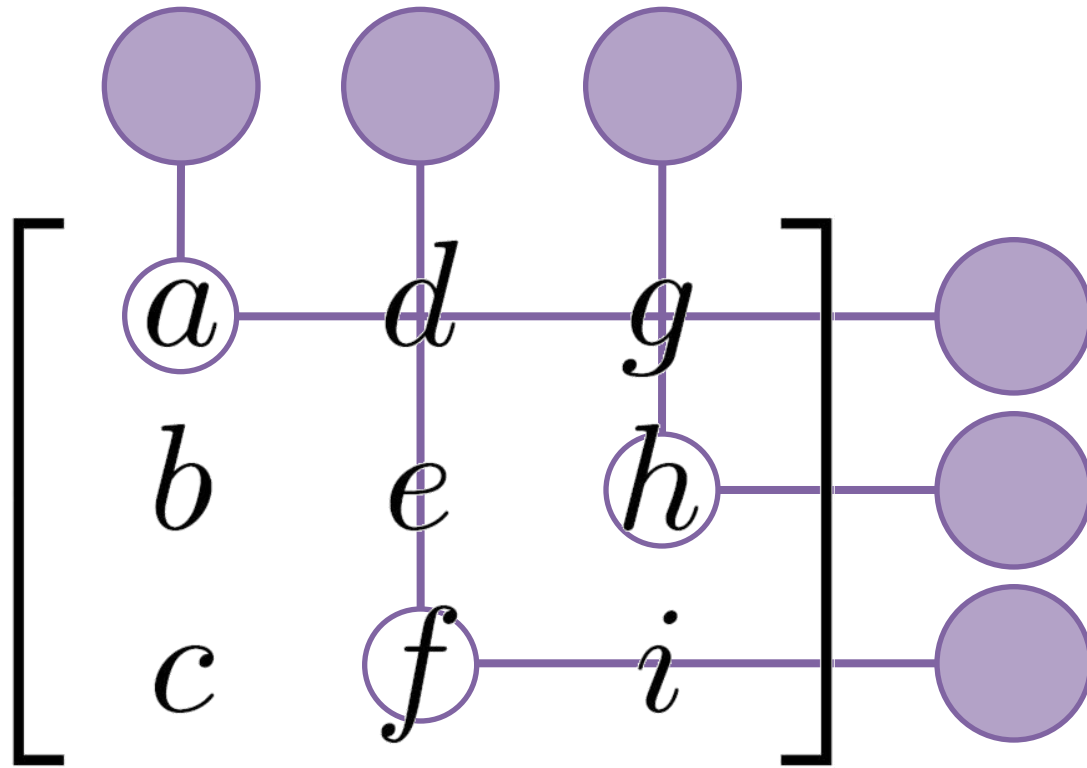
Pr [one per slot] =

# A Transition Probability



$$\Pr [\text{one per slot}] = aei +$$

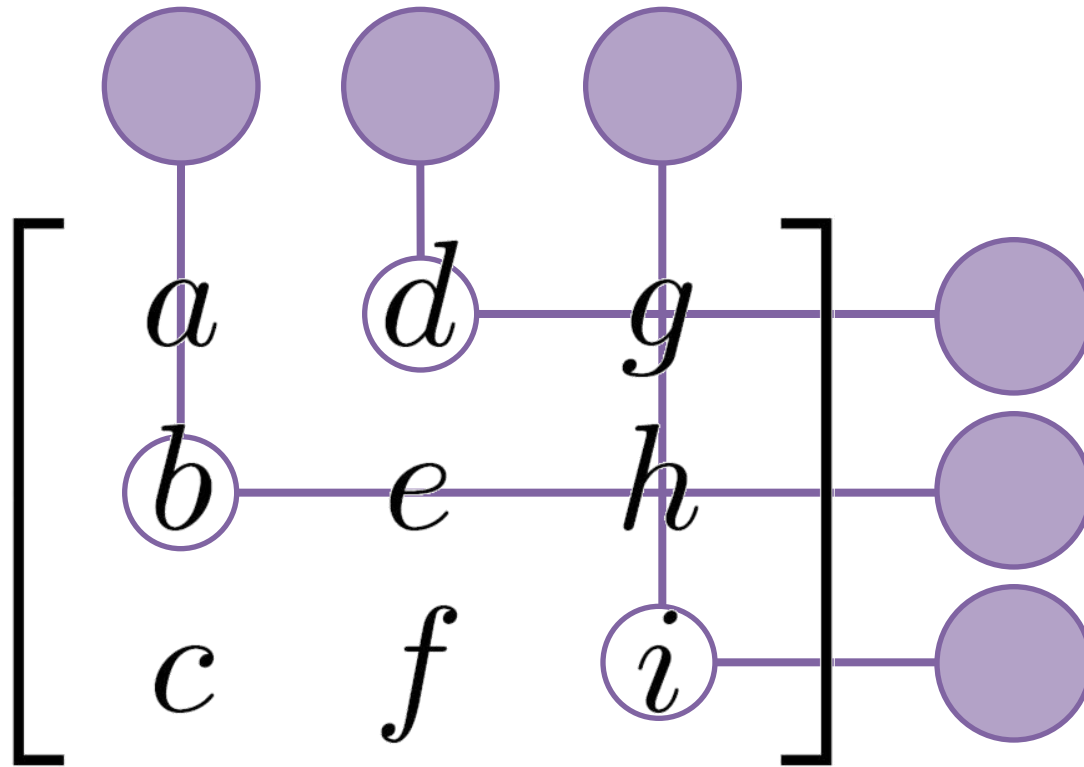
# A Transition Probability



$$\Pr [\text{one per slot}] = aei + afh$$

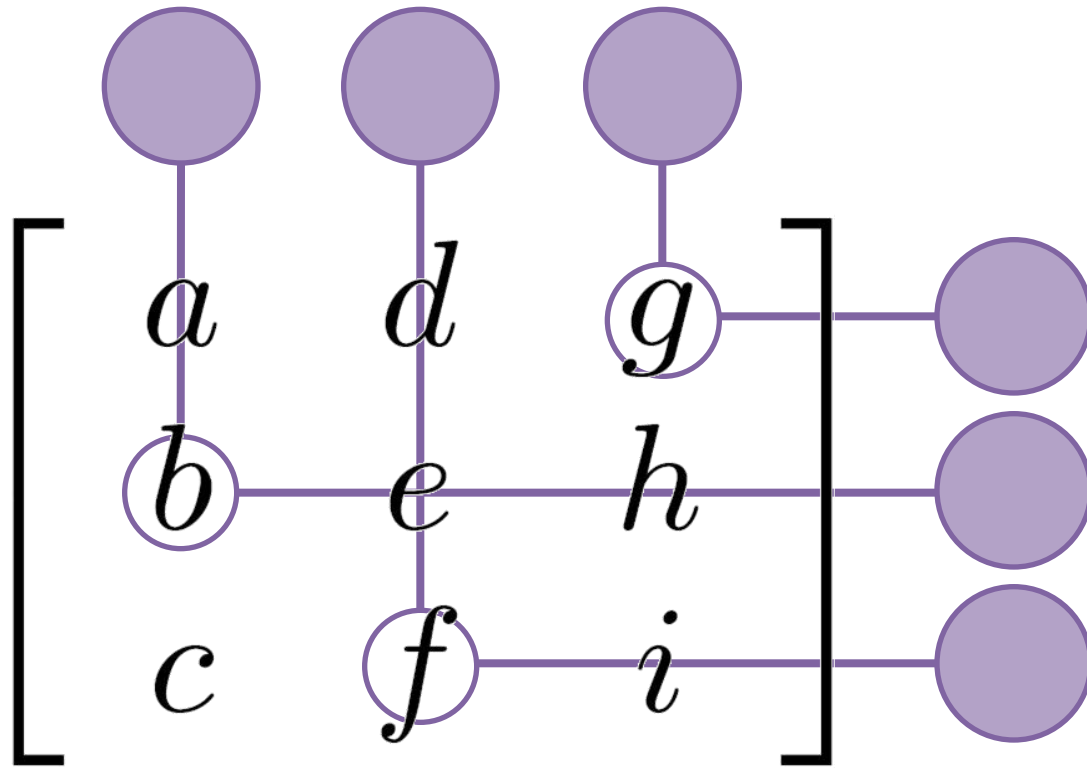


# A Transition Probability



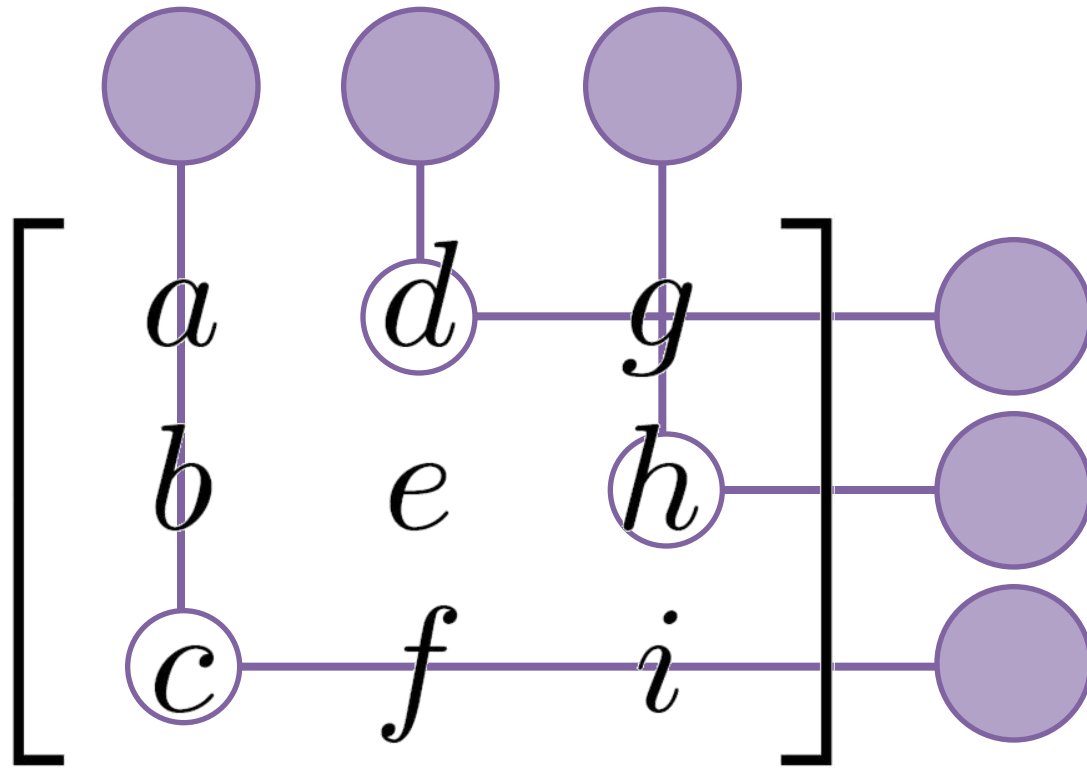
$$\Pr [\text{one per slot}] = aei + afh + bdi$$

# A Transition Probability



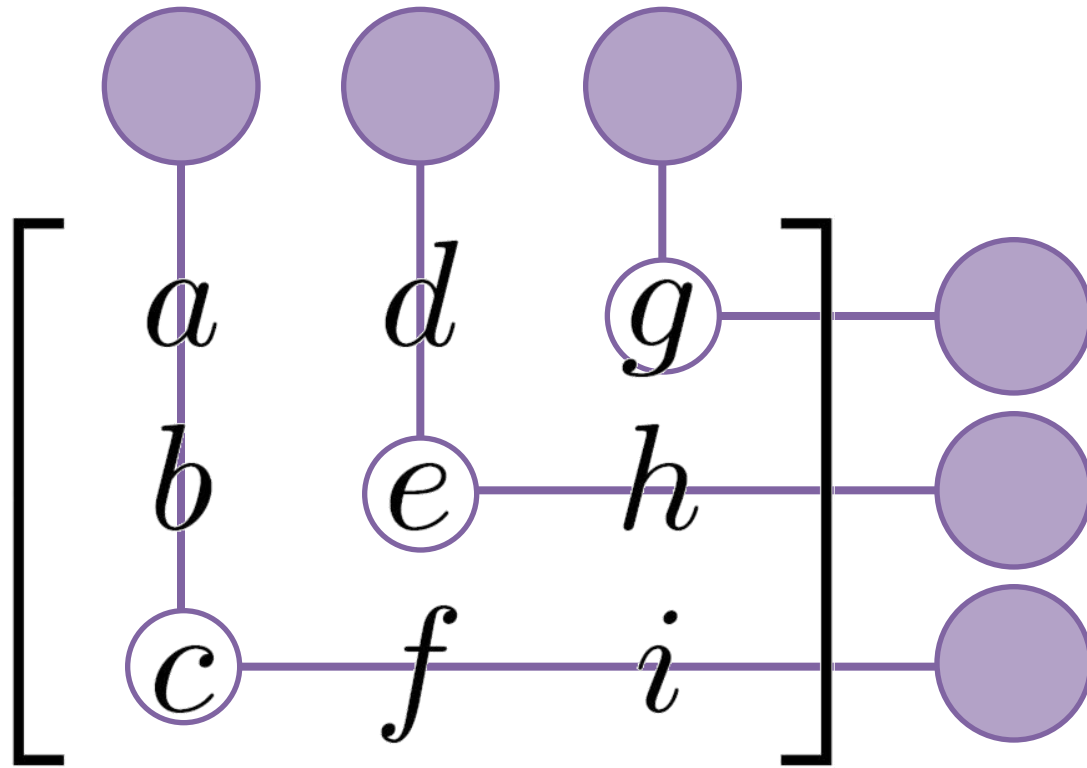
$$\Pr [\text{one per slot}] = aei + afh + bdi + bfg$$

# A Transition Probability



$$\Pr [\text{one per slot}] = aei + afh + bdi + bfg + cdh$$

# A Transition Probability



$$\begin{aligned} \Pr [\text{one per slot}] &= aei + afh + bdi + bfg + cdh + ceg \\ &= \text{perm}(M) \end{aligned}$$

# Probabilities for Classical Analogue

$$\Pr [\text{one per slot} \rightarrow \text{one per slot}] = \sum_{\sigma \in S_n} \prod_{i=1}^n M_{\sigma(i), i}$$

$$\text{perm} (M) = \sum_{\sigma \in S_n} \prod_{i=1}^n M_{\sigma(i), i}$$

$$\det (M) = \sum_{\sigma \in S_n} \text{sgn} (\sigma) \prod_{i=1}^n M_{\sigma(i), i}$$

# Probabilities for Classical Analogue

- What about other transitions?

Diagram illustrating a transition from a 3x3 matrix to a permutation matrix. The matrix is:

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Two purple circles are positioned to the right of the matrix, followed by a question mark. The permutation matrix is:

$$\text{perm} \begin{bmatrix} a & g \\ b & h \end{bmatrix}$$

Diagram illustrating a transition from a 3x3 matrix to a permutation matrix. The matrix is:

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Two purple circles are positioned to the right of the matrix, followed by a question mark. The permutation matrix is:

$$\frac{1}{2} \text{perm} \begin{bmatrix} a & a & g \\ b & b & h \\ b & b & h \end{bmatrix}$$

# Configuration Transitions

Transition matrix  
for one ball

$$\begin{array}{c}
 \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} \\
 \left[ \begin{array}{ccc} a & d & g \\ b & e & h \\ c & f & i \end{array} \right] \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \\
 m=3, n=1
 \end{array}$$

Transition matrix for two-ball configurations

$$\begin{array}{c}
 \begin{array}{cccccc} \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet \\ \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet \\ \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet & \bullet \bullet \end{array} \\
 \left[ \begin{array}{ccc} a^2 & d^2 & g^2 \\ b^2 & e^2 & h^2 \\ c^2 & f^2 & i^2 \\ 2ab & 2de & 2gh \\ 2ac & 2df & 2gi \\ 2bc & 2ef & 2hi \end{array} \right] \begin{array}{ccc} ad & ag & dg \\ be & bh & eh \\ cf & ci & fi \\ ae + bd & ah + bg & dh + eg \\ af + cd & ai + cg & ei + fh \\ bf + ce & bi + ch & di + fg \end{array} \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \\
 m=3, n=2
 \end{array}$$

# Classical Model Summary

- $n$  identical balls
- $m$  slots
- Choose start **configuration**
- Choose stochastic **transition matrix**  $M$
- Move each ball as per  $M$
- Look at resulting configuration

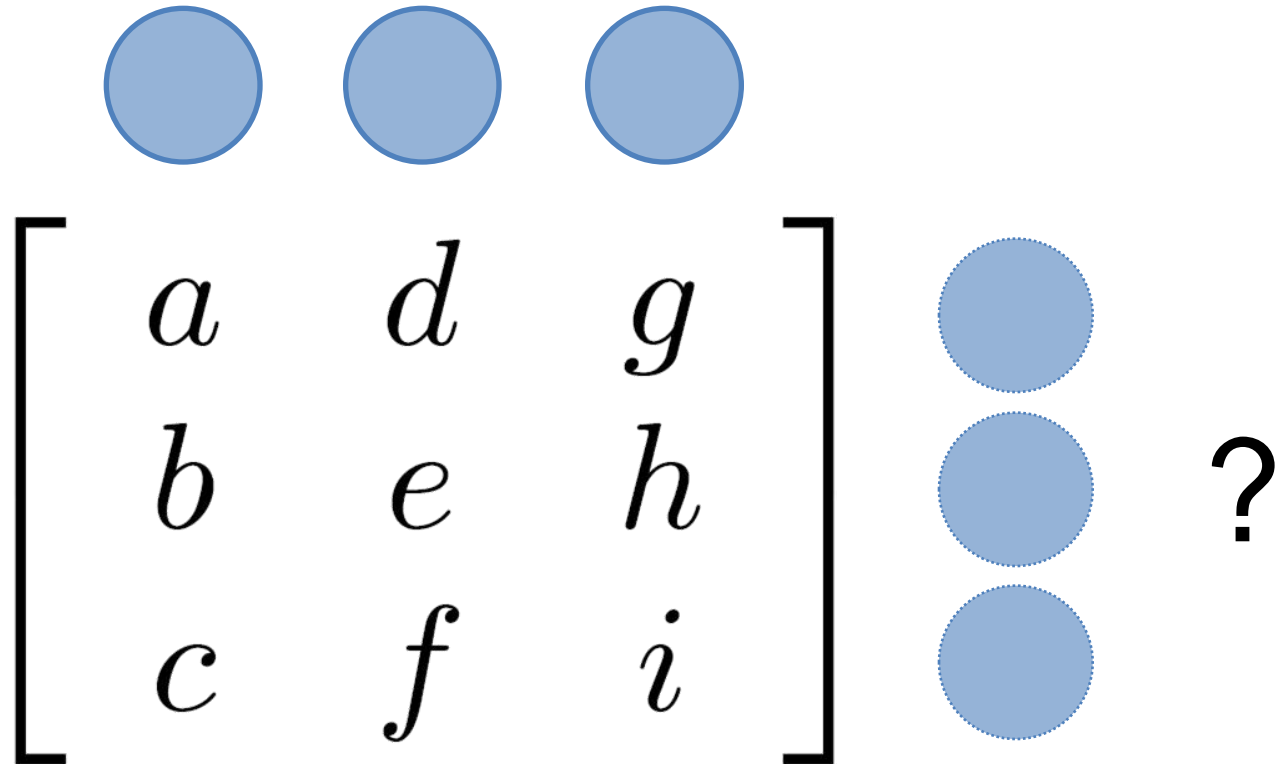


# Quantum Model

# Quantum Particles

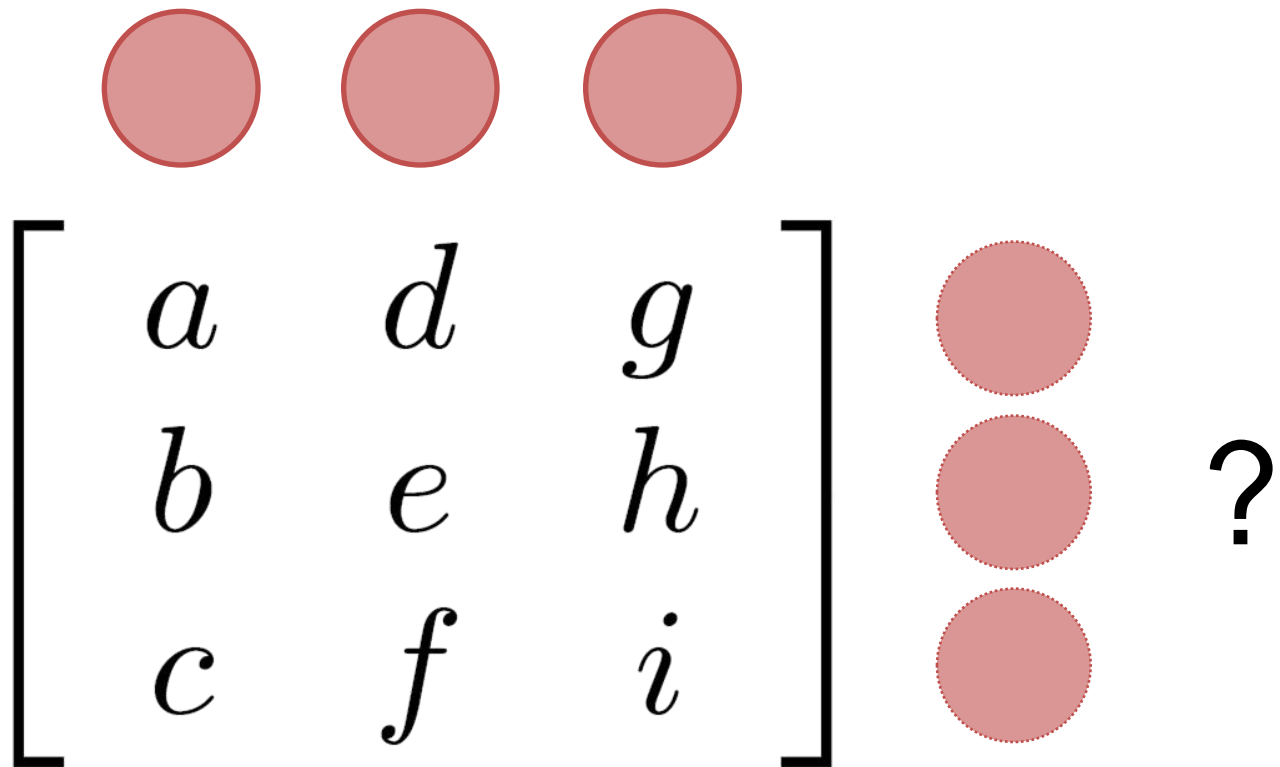
- Two types of particle: **Bosons** and **Fermions**

# Identical Bosons



$$\begin{aligned} \text{Am [one per slot]} &= aei + afh + bdi + bfg + cdh + ceg \\ &= \text{perm}(M) \\ \text{Pr [one per slot]} &= |\text{perm}(M)|^2 \end{aligned}$$

# Identical Fermions



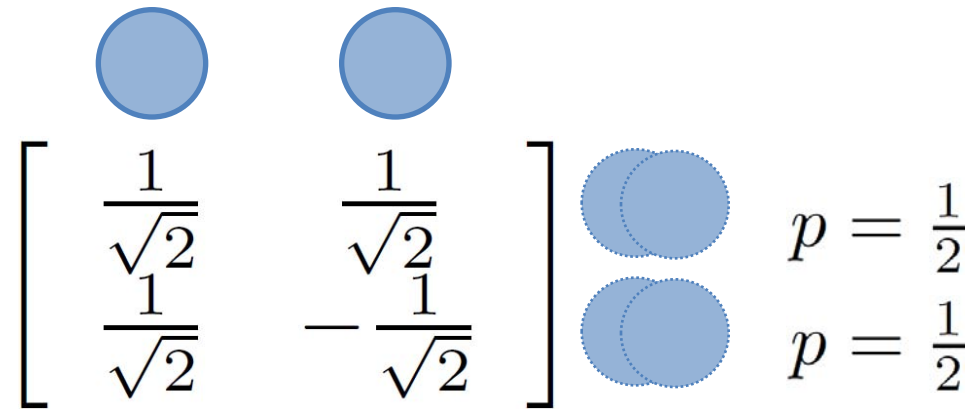
$$\begin{aligned} \text{Am [one per slot]} &= aei - afh - bdi + bfg + cdh - ceg \\ &= \det(M) \end{aligned}$$

$$\text{Pr [one per slot]} = |\det(M)|^2$$

# Algebraic Formalism

- Modes are single-particle basis states
  - Variables  $x_1, \dots, x_m$
- Configurations are multi-particle basis states
  - Monomials  $x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} / \sqrt{a_1! \cdots a_m!}$
- Identical **bosons** commute
  - $x_i x_j = x_j x_i$
- Identical **fermions** anticommute
  - $x_i x_j = -x_j x_i$
  - $x_i^2 = 0$

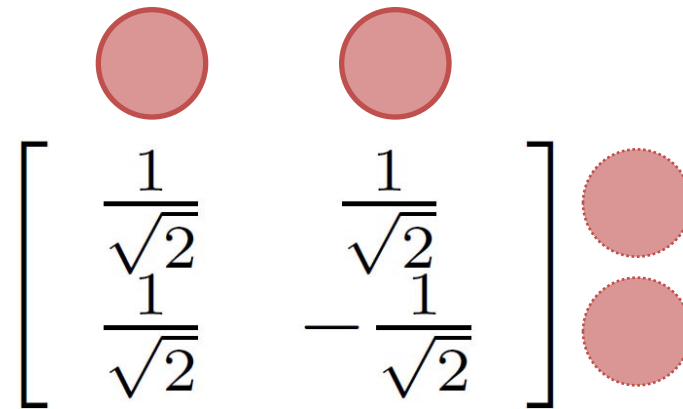
# Example: Hadamarding Bosons



$$\begin{aligned}
 xy &\Rightarrow \left( \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \right) \left( \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \right) \\
 &= \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2}} x^2 \right] - \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2}} y^2 \right]
 \end{aligned}$$

Hong-Ou-Mandel dip

# Example: Hadamarding Fermions

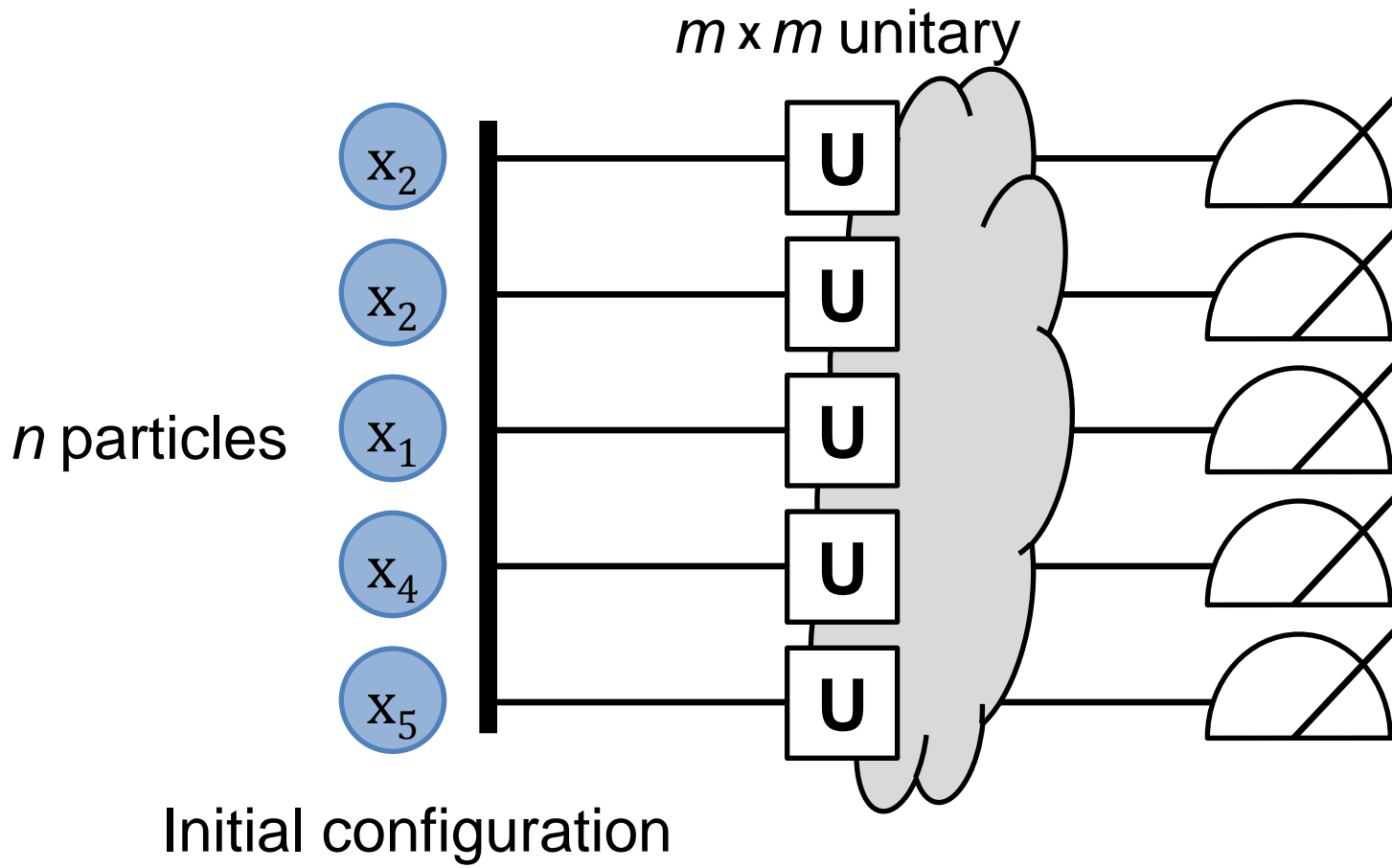


The diagram shows a Hadamard gate represented by a large square bracket containing two columns of terms. Above the first column is a solid red circle, and above the second column is another solid red circle. To the right of the bracket are two vertically stacked red circles, the top one being dashed and the bottom one solid.

$$\left[ \begin{array}{cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array} \right]$$

$$\begin{aligned} xy &\Rightarrow \left( \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \right) \left( \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \right) \\ &= \frac{1}{2}x^2 + \frac{1}{2}xy - \frac{1}{2}yx + \frac{1}{2}y^2 \\ &= xy \end{aligned}$$

# Definition of Model





Complexity

# Complexity Comparison

Particle:			
Function:			
Matrix:			
Compute probability:			
Sample:			

Adaptive  $\rightarrow$  BQP  
[KLM '01]

# Bosons Have the Hard Job

- **Fermions: Easy**
  - Det is in P
  - Doable in P [Valiant '01]
- **Classical particles: Easy**
  - Perm is #P-complete!
  - Perm approximable for  $\geq 0$  matrices [JSV '01]
- **Bosons: Hard**
  - With adaptive measurements, get BQP [KLM '01]
  - Not classically doable, even approximately [AA '10 in prep]

# Bosons are Hard: Proof

- Classically simulate identical bosons

Approx counting

- Using NP oracle, estimate  $|\text{perm}(M)|^2$

Reductions

- Compute permanent in  $\text{BPP}^{\text{NP}}$

Perm is #P-complete

- $\text{P}^{\#\text{P}}$  lies within  $\text{BPP}^{\text{NP}}$

Toda's Theorem

- Polynomial hierarchy collapses

# *Approximate* **Bosons** are Hard: Proof

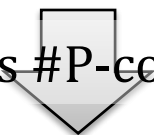
- Classically *approximately* simulate identical **bosons**

Approx counting  


- Using NP oracle, estimate  $|\text{perm}(M)|^2$  of random  $M$  with high probability

Random self-reducibility  
+ conjectures  


- Compute permanent in  $\text{BPP}^{\text{NP}}$

Perm is #P-complete  


- $\text{P}^{\#\text{P}}$  lies within  $\text{BPP}^{\text{NP}}$

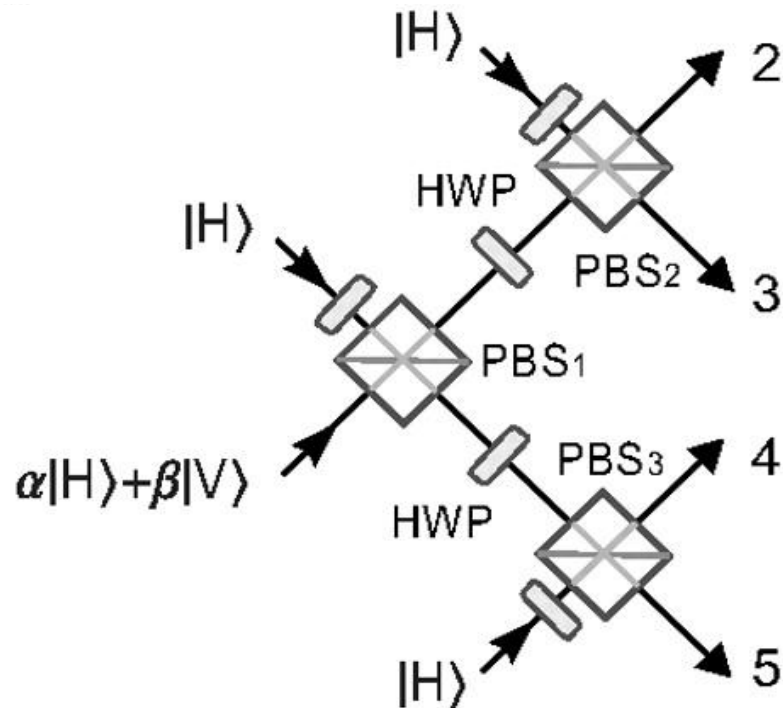
Toda's Theorem  


- Polynomial hierarchy collapses

# Experimental Prospects

# Linear Optics

- **Photons** and half-silvered mirrors



- Beamsplitters + phaseshifters are universal

# Challenge: Do These Reliably

- Encode values into mirrors
- Generate single photons
- Have photons hit mirrors at same time
- Detect output photons



# Proposed Experiment

- Use  $m=20, n=10$
- Choose  $U$  at random
- Check by brute force!

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6.845 Quantum Complexity Theory  
Fall 2010

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