Inventory Management
Probabilistic Demand

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Assumptions: Probabilistic Demand

- **Demand**
  - Constant vs **Variable**
  - Known vs **Random**
  - **Continuous** vs Discrete

- **Lead time**
  - Instantaneous
  - **Constant** or Variable (deterministic/stochastic)

- **Dependence of items**
  - **Independent**
  - Correlated
  - Indentured

- **Review Time**
  - **Continuous**
  - Periodic

- **Number of Echelons**
  - **One**
  - Multi (>1)

- **Capacity / Resources**
  - **Unlimited**
  - Limited / Constrained

- **Discounts**
  - **None**
  - All Units or Incremental

- **Excess Demand**
  - None
  - **All orders are backordered**
  - **All orders are lost**

- **Substitution**

- **Perishability**
  - **None**
  - Uniform with time

- **Planning Horizon**
  - Single Period
  - Finite Period
  - **Infinite**

- **Number of Items**
  - **One**
  - Many

- **Form of Product**
  - **Single Stage**
  - Multi-Stage
Key Questions

What are the questions I should ask to determine the type of inventory control system to use?

- How important is the item?
- Should review be periodic or continuous?
- What form of inventory policy should I use?
- What cost or service objectives should I set?
How important is the item?

- Standard ABC analysis
  - **A Items**
    - Very few high impact items are included
    - Require the most managerial attention and review
    - Expect many exceptions to be made
  - **B Items**
    - Many moderate impact items (sometimes most)
    - Automated control w/ management by exception
    - Rules can be used for A (but usually too many exceptions)
  - **C Items**
    - Many if not most of the items that make up minor impact
    - Control systems should be as simple as possible
    - Reduce wasted management time and attention
    - Group into common regions, suppliers, end users

- But – these are arbitrary classifications
### Continuous or Periodic Review?

#### Periodic Review
- Know stock level only at certain times
- Review periods are usually scheduled and consistent
- Ordering occurs at review

#### Pros / Cons
- Coordination of replenishments
- Able to predict workload
- Forces a periodic review

#### Continuous Review
- Is continuous really continuous?
- Transactions reporting
- Collecting information vs. Making decision

#### Pros / Cons
- Replenishments made dynamically
- Cost of equipment
- Able to provide same level of service with less safety stock

#### Notation
- $s =$ Order Point
- $S =$ Order-up-to Level
- $Q =$ Order Quantity
- $R =$ Review Period
- $L =$ Order Lead Time
- $IOH =$ Inventory on Hand
- $IP =$ Inventory Position
- $(IOH) + (Inv \; On \; Order) - (Backorders) - (Committed)$
What form of inventory policy?

Continuous Review (R=0)

- **Order-Point, Order-Quantity (s, Q)**
  - Policy: Order Q if IP ≤ s
  - Two-bin system

- **Order-Point, Order-Up-To-Level (s, S)**
  - Policy: Order (S-IP) if IP ≤ s
  - Min-Max system
What form of inventory policy?

Periodic Review (R>0)

- **Order-Up-To-Level (R, S)**
  - Policy: Order S-IP every R time periods
  - Replenishment cycle system

- **Hybrid (R, s, S) System**
  - Policy: Order S-IP if IP ≤ s every R time periods, if IP>s then do not order
  - General case for many policies
What form of inventory policy?

No hard and fast rules, but some rules of thumb

<table>
<thead>
<tr>
<th>Type of Item,</th>
<th>Continuous Review</th>
<th>Periodic Review</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Items</td>
<td>(s, S)</td>
<td>(R, s, S)</td>
</tr>
<tr>
<td>B Items</td>
<td>(s, Q)</td>
<td>(R, S)</td>
</tr>
<tr>
<td>C Items</td>
<td>Manual ~ (R, S)</td>
<td></td>
</tr>
</tbody>
</table>

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Determining $s$ in $(s,Q)$ System

- Coverage over lead time
  - Expected demand over lead time
  - Safety (buffer stock)

- Procedure:
  - Find Safety Stock (SS) by specifying a $k$
  - Find $s$ by adding SS to expected demand over leadtime

\[ s = x_L + k \sigma_L \]

- Reorder Point
- Forecast demand over the lead time
- Safety Stock
  \[ k = \text{SS factor} \]
  \[ \sigma_L = \text{RMSE} \]

Parameters depend on cost & service objectives
What cost and service objectives?

1. Common Safety Factors Approach
   - Simple, widely used method
   - Apply a common metric to aggregated items

2. Cost Minimization Approach
   - Requires costing of shortages
   - Find trade-off between relevant costs

3. Customer Service Approach
   - Establish constraint on customer service
   - Definitions in practice are fuzzy
   - Minimize costs with respect to customer service constraints

4. Aggregate Considerations
   - Weight specific characteristic of each item
   - Select characteristic most “essential” to firm
Framework for (s, Q) Systems

- **Cycle Stock**
  - Determine best Q
  - Usually from EOQ

- **Safety Stock**
  - Pick type of cost or service standard
    - If service, then use decision rule for setting k
    - If cost, then minimize total relevant costs to find k
  - Calculate safety stock as \( k\sigma_L \)

- **Total Cost:**

\[
TC = vD + A\left(\frac{D}{Q}\right) + vr\left(\frac{Q}{2} + k\sigma_L\right) + C_{\text{StockOutType}}P[\text{StockOutType}]
\]
### Framework for (s,Q) Systems

<table>
<thead>
<tr>
<th>Stockout Types</th>
<th>Key Element</th>
<th>Cost</th>
<th>Service</th>
</tr>
</thead>
<tbody>
<tr>
<td>Event based</td>
<td>Probability of a stock out event</td>
<td>$B_1(\text{Prob}<a href="D/Q">\text{SO}</a>)$</td>
<td>$P_1 = 1 - \text{Prob}[\text{SO}]$</td>
</tr>
<tr>
<td># of Units Short</td>
<td>Expected # units short</td>
<td>$(B_2v)(\sigma L_G_u(k))(D/Q)$</td>
<td>$P_2 = \text{ItemFillRate}$ = 1 - $(\sigma L_G_u(k)/Q)$</td>
</tr>
<tr>
<td>Units Short per Time</td>
<td>Expected duration time for each unit stocked out</td>
<td>$(B_3v)(\sigma L_G_u(k)d_{SO})(D/Q)$ Where $d_{SO} = \text{avg duration of stockout}$</td>
<td></td>
</tr>
<tr>
<td>Line Items Short</td>
<td>Expected number of lines shorted</td>
<td>$(B_4v)(\sigma L_G_u(k)/z)(D/Q)$ where $z = \text{avg items / order}$</td>
<td></td>
</tr>
</tbody>
</table>
Cycle Service Level (CSL or $P_1$)

**Cycle Service Level**
- Probability of no stockouts per replenishment cycle
- Equal to one minus the probability of stocking out
- $= 1 - P[\text{Stockout}] = 1 - P[x_L > s] = P[x_L \leq s]$
Finding $P[\text{Stockout}]$

Forecast Demand $\sim$ iid $N(x_L=500, \sigma_{err}=50)$

- Probability of NOT stocking out during an order cycle

- Probability of stocking out during an order cycle

Forecasted Demand ($x_L$)
Cumulative Normal Distribution

Forecast Demand $\sim$ iid $N(\mu=500, \sigma_{\text{err}}=50)$

Probability of Stockout $= 1 - \text{CSL}$

Cycle Service Level or $P_1$

Reorder Point

Cycle Service Level or $P_1$
Finding CSL from a given k

Using a Table of Cumulative Normal Probabilities . . .

<table>
<thead>
<tr>
<th>K</th>
<th>0.00</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.5000</td>
<td>0.5040</td>
<td>0.5080</td>
<td>0.5120</td>
<td>0.5160</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5398</td>
<td>0.5438</td>
<td>0.5478</td>
<td>0.5517</td>
<td>0.5557</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5793</td>
<td>0.5832</td>
<td>0.5871</td>
<td>0.5910</td>
<td>0.5948</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6179</td>
<td>0.6217</td>
<td>0.6255</td>
<td>0.6293</td>
<td>0.6331</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6554</td>
<td>0.6591</td>
<td>0.6628</td>
<td>0.6664</td>
<td>0.6700</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6915</td>
<td>0.6950</td>
<td>0.6985</td>
<td>0.7019</td>
<td>0.7054</td>
</tr>
</tbody>
</table>

If I select a k=0.42

From SPP (Table B.1 pp 724-734)
- Select k factor (first column)
- Prob[Stockout] = value in the $p_{u\geq}(k)$ column
- CSL = 1 - $p_{u\geq}(k)$

In Excel, use the function
- CSL=NORMDIST(s, x_L, σ_L, 1) where s = x_L+kσ_L
- CSL=NORMSDIST(k)

. . . then my Cycle Service Level is this value.
k Factor versus Cycle Service Level

![Graph showing the relationship between K factor and service level](image)

Figure by MIT OCW.
Example: Setting SS and s

**Given**
- Average demand over time is considered constant
- Forecast of demand is 13,000 units a year ~ iid Normal
- Lead time is 2 weeks
- RMSE of the forecast = 1,316 units per year
- EOQ = 228 units (A=50 $/order, r=10\%, v=250 $/item)

**Find**
- Safety stock and reorder point, $s$, for the following cycle service levels:
  - CSL=.80
  - CSL=.90
  - CSL=.95
  - CSL=.99
Quick Aside on Converting Times

How do I convert expected values and variances of demand from one time period to another?

Suppose we have two periods to consider:
- $S =$ Demand over short time period (e.g., week)
- $L =$ Demand over long time period (e.g., year)
- $n =$ Number of short periods within a long (e.g., 52)

Converting from Long to Short
- $E[S] = E[L]/n$
- $\text{VAR}[S] = \text{VAR}[L]/n$ so that $\sigma_S = \sigma_L/\sqrt{n}$

Converting from Short to Long
- $E[L] = nE[S]$
- $\text{VAR}[L] = n\text{VAR}[S]$ so that $\sigma_L = \sqrt{n} \sigma_S$
Item Fill Rate \((P_2)\) Metric

- Item Fill Rate \((P_2)\)
  - Fraction of demand filled from IOH
  - Need to find the expected number of items that I will be short for each cycle
    - Expected Units Short \(E[US]\)
    - Expected Shortage per Replenishment Cycle (ESPRC)
  - More difficult than CSL – need to find a partial expectation for units short

\[
\text{Fill Rate} = \frac{\text{Order Quantity} - E[\text{Units Short}]}{\text{Order Quantity}}
\]
Finding Expected Units Short

Find the expected number of units short, \( E[US] \), during a replenishment cycle.

Use Loss Function – widely used in inventory theory.

\( L(k) = \text{expected amount that random variable } X \text{ exceeds a given threshold value, } k. \)

Interpretation:

If my demand is \( \sim U(1,8) \) and I have a safety stock of 5 then I can expect to be short 0.75 units each service cycle.

What is \( L[k] \) if \( k=5 \)?
Finding Expected Units Short

Consider both continuous and discrete cases
Looking for expected units short per replenishment cycle.

\[ E[US] = \sum_{x=k}^{\infty} (x - k) p(x) \]
\[ E[US] = \int_{k}^{\infty} (x_o - k) f_x(x_o) dx_o \]

For normal distribution, \( E[US] = \sigma_L G(k) \)
Where \( G(k) = \) Unit Normal Loss Function

In SPP,
\( G(k) = G_u(k) = f_x(x_o) - k \cdot \text{Prob}[x_o \geq k] \)
Derived in SPP p. 721, in tables B.1

In Excel,
\( \text{NORMDIST}(k,0,1,0) - k(1 - \text{NORMDIST}(k,0,1,1)) \)
Item Fill Rate (IFR or P₂)

Procedure: Relate k to desired IFR
- Find k that satisfies rule
  - Solve for G[k]
  - Use table or Excel to find k
- Find reorder point s
  - \( s = x_L + k\sigma_L \)

Example
- Average demand over time is considered constant
- Forecast of demand is 13,000 units a year ~ iid Normal
- Lead time is 2 weeks
- RMSE of the forecast = 1,316 units per year
- EOQ = 228 units (A=50 $/order, r=10%, v=250 $/item)

Find
- Safety stock and reorder point, s, for the following item fill rates:
  - IFR=.80, .90, .95, and 0.99

\[
IFR = \frac{Q - E[US]}{Q} = 1 - \frac{E[US]}{Q}
\]

\[
IFR = 1 - \frac{\sigma_L G[k]}{Q}
\]

\[
G[k] = \frac{Q}{\sigma_L} (1 - IFR)
\]
Compare CSL versus IFR

IFR usually much higher than CSL for same SS
IFR depends on both s and Q while CSL is independent of all product characteristics
Q determines the number of exposures for an item

<table>
<thead>
<tr>
<th>Pct</th>
<th>SS CSL</th>
<th>SS IFR</th>
</tr>
</thead>
<tbody>
<tr>
<td>99%</td>
<td>601</td>
<td>513</td>
</tr>
<tr>
<td>95%</td>
<td>423</td>
<td>348</td>
</tr>
<tr>
<td>90%</td>
<td>330</td>
<td>252</td>
</tr>
<tr>
<td>80%</td>
<td>217</td>
<td>148</td>
</tr>
</tbody>
</table>
Consider total relevant costs

- Order Costs – no change from EOQ
- Holding Costs – add in Safety Stock
- StockOut Costs product of:
  - Cost per stockout event \( B_1 \)
  - Number of replenishment cycles
  - Probability of a stockout per cycle

\[
TRC = Order Costs + Holding Costs + StockOut Costs \\
TRC = A \left( \frac{D}{Q} \right) + \left( \frac{Q}{2} + k\sigma_L \right) vr + B_1 \left( \frac{D}{Q} \right) p_{u\geq}(k)
\]

Solve for \( k \) that minimizes total relevant costs

- Use solver in Excel
- Use decision rules
Cost per Stockout Event \((B_1)\)

**Decision Rule**

- If Eqn 7.19 is true
  - Set \(k\) to lowest allowable value (by mgmt)
- Otherwise set \(k\) using Eqn 7.20

\[
(Eqn7.19) \quad \frac{DB_1}{\sqrt{2\pi Qv\sigma_L r}} < 1
\]

\[
(Eqn7.20) \quad k = \sqrt{2 \ln \left( \frac{DB_1}{\sqrt{2\pi Qv\sigma_L r}} \right)}
\]
Cost per Unit Short \((B_2)\)

- Consider total relevant costs
  - Order Costs – no change from EOQ
  - Holding Costs – add in Safety Stock
  - StockOut Costs product of:
    - Cost per item stocked out \((B_2)\)
    - Estimated number units short
    - Number of replenishment cycles

\[
TRC = OrderCosts + HoldingCosts + StockOutCosts
\]

\[
TRC = A \left( \frac{D}{Q} \right) + \left( \frac{Q}{2} + k\sigma_L \right)vr + B_2v\sigma_L G_u(k) \left( \frac{D}{Q} \right)
\]

- Solve for \(k\) that minimizes total relevant costs
  - Use solver in Excel
  - Use decision rules
Cost per Unit Short ($B_2$)

**Decision Rule**

- If Eqn 7.22 is true
  - Set $k$ to lowest allowable value (by mgmt)
- Otherwise set $k$ using Eqn 7.23

\[
(Eqn 7.22) \quad \frac{Q_r}{DB_2} > 1
\]

\[
(Eqn 7.23) \quad p_{u \geq k} = \frac{Q_r}{DB_2}
\]
Example

You are setting up inventory policy for a Class B item. The annual demand is forecasted to be 26,000 units with an annual historical RMSE +/- 2,800 units. The replenishment lead time is currently 4 weeks. You have been asked to establish an (s,Q) inventory policy.

Other details: It costs $12,500 to place an order, total landed cost is $750 per item, holding cost is 10%. Items come in cases of 100 each.

What is my policy, safety stock, and avg IOH if . . .

1. I want to have a CSL of 95%?
2. I want to achieve an IFR of 95%?
3. I estimate that the cost of a stockout per cycle is $50,000?
4. I estimate that the cost of a stockout per item is $75?
Questions?
Comments?
Suggestions?