Problem a1

Answers may vary, but anything along the lines of, we expect this to be a great class, will be accepted.

Problem a2

a2-1)

Wind Turbine
- Boundary: Wind turbine blades, connection to the earth, connection to power grid
- Inputs: Wind velocity (speed+direction), atmospheric pressure/temperature/humidity, disturbances
- Outputs: Power, Volume/Height, Noise

Cable Stayed Bridge
- Boundary: Connection to street, connection to ground, entire surface for wind induced vibrations
- Inputs: Traffic loads and variation, external loads such as wind or water
- Outputs: Size, cost

Plug-in-hybrid Electric Car
- Boundary: Tire surface to road, the plug, fuel tank nozzle
- Inputs: Number of passengers, voltage from wall, gas, driving conditions-break or gas pedal use
- Outputs: Fuel economy, driving performance, cost, pollution, noise

Submarine
- Boundary: Submarine structure (hull)
- Inputs: water (ballasts), communications signals (electromagnetic), sensing signals (SONAR)
- Outputs: water (ballasts), communications signals, SONAR, propelling forces, weapons

a2-2)
(a) Wind Turbine.

(b) Cable stayed bridge.

(c) Plug-in-hybrid electric car.

(d) Submarine.

Figure 1: System component descriptions.
(a) Sketch and interpret the following function:

\[ f(x) = x_1^2 + 5x_2^2 - 2x_1^3x_2 + x_1^4 + 2x_2^4 \quad \forall x \in [-5, 5] \]  
(1)

\[ \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 6x_1^2x_2 + 4x_1^3 \\ 10x_2 - 2x_1^3 + 8x_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  
(2)

This shows \( x = (0, 0) \) is the only stationary point, to check to see if this point is a minimum, maximum, or saddle point, the Hessian must be checked.

\[ H(x) = \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 - 12x_1x_2 + 12x_1^2 & -6x_1^2 \\ -6x_1^2 & 10 + 24x_2 \end{bmatrix} \]  
(4)

\[ H(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}. \]  
(5)
$H(0,0)$ is positive definite so the point $x = (0,0)$ is a local minimizer of $f(x)$. From the plot, Figure 3, it is clear that $x = (0,0)$ is actually the global minimizer of $f(x)$.

![Figure 3: $f(x) = x_1^2 + 5x_2^2 - 2x_1^3x_2 + x_1^4 + 2x_2^4$ $\forall x \in [-5,5]$.](image)

(b) Same as above, but add the constraint,

$$g(x) = (x_1 - 3)^2 + 2x_2^2 + 3x_1x_2 - 2 \leq 0$$  

(6)

$g(0,0) = 7$, so the constraint is violated at the local minimum $x = (0,0)$. This means the constraint will be active at the constrained minimum, $g(x^0) = 0$. The minimum is now at, $x^* = [1.2608, -0.3278]^T$. See Figure 4.

**Problem a3**

a3-1)

Given the function:

$$f(x) = x_1^4 - x_1^2x_2 + x_2^2 + \frac{1}{2}x_1^2$$  

(7)
Compute the gradient and Hessian of \( f(x) \). Show that \( x^* = (0, 0) \) is the only local minimize of this function and that the Hessian matrix at that point is positive definite.

\[
\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1^2 - 2x_1x_2 + x_1 \\ -x_1^2 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}
\]

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} \pm \frac{\sqrt{3}}{3} \\ -\frac{1}{6} \end{bmatrix} \tag{9}
\]

This shows \( x = (0, 0) \) is the only stationary point. To show that it is a local minimum, we must check the Hessian, \( H(0, 0) \):

\[
H(x) = \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 12x_1^2 - 2x_2 + 1 & -2x_1 \\ -2x_1 & 2 \end{bmatrix} \tag{10}
\]

\[
H(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \tag{11}
\]

The Hessian is positive definite, so \( x^* = (0, 0) \) is a local minimizer, and as \( x^* = (0, 0) \) is the only stationary point it is accordingly the only local minimizer.
a3-2)
Make a contour plot of the objective value, \( f(x) \), versus the design variables \( x_1, x_2 \) and verify the local minimum graphically.

![Contour Plot](image)

Figure 5: \( f(x) = x_1^4 - x_1^2 x_2 + x_2^2 + \frac{1}{2} x_1^2 \) \( \forall x_1 \in [-5, 5], x_2 \in [-20, 20] \).

a3-3)
Show that the function,

\[
f(x) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2,
\]

has only one stationary point, and that it is neither a maximum or minimum, but a saddle point.

\[
\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
(13)
\]

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
(14)
\]

This shows there is only one stationary point for \( f(x) \) and it’s at \( x = (1, 1) \). To classify it as a maximum, minimum, or saddle point, we need to check the Hessian,

\[
H(x) = \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}
\]

\[
(15)
\]
The eigenvalues of the Hessian are $\lambda_1 = -0.5311$ and $\lambda_2 = 7.5311$, and one is positive, and one is negative. Accordingly, $H(1,1)$ is indefinite and $x = (1, 1)$ is a saddle point. See Figure 6.

![Figure 6: $f(x) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$ for all $x \in [-5, 5]$.](image)

a3-4) How many stationary points does the function,

$$ f(x) = \frac{1}{3}x_1^3 + x_1x_2 + \frac{1}{2}x_2^2 + 2x_2 - 5, \quad (16) $$

have? Classify all of the stationary points as either maximum, minimum, or saddle points.

$$ \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2 \\ x_1 + x_2 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (17) $$

$$ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad (18) $$

To classify the two stationary points, the Hessian must be checked:

$$ H(x) = \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 1 \\ 1 & 1 \end{bmatrix} \quad (19) $$

$$ H(2, -4) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \quad (20) $$

$$ H(-1, -1) = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \quad (21) $$
$H(2, -4)$ is positive definite, so $\mathbf{x}^* = [2, -4]^T$ is a local minimum. $H(-1, -1)$ is indefinite, so $\mathbf{x}^* = [-1, -1]^T$ is a saddle point. See Figure 7.

Figure 7: \( f(\mathbf{x}) = \frac{1}{3}x_1^3 + x_1x_2 + \frac{1}{2}x_2^2 + 2x_2 - 5 \) \( \forall x_1 \in [-5, 5], x_2 \in [-10, 5] \).
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