Question 1

I first prove that if \( \neg P \lor Q \in \Gamma^* \), then \( P \in \Gamma^* \) or \( Q \in \Gamma^* \). I do this by proving the contrapositive — i.e., that if it is not the case that \( P \in \Gamma^* \) or \( Q \in \Gamma^* \), then it is not the case that \( \neg P \lor Q \in \Gamma^* \).

1. Suppose it is not the case that \( P \in \Gamma^* \) or \( Q \in \Gamma^* \).

2. Then \( \neg P \lor Q \in \Gamma^* \) implies \( \neg P \lor Q \in \Gamma^* \) (by 6.4.11(a)).

3. So \( \{ \neg P, \neg Q \} \) \( \in \Gamma^* \).

4. Now, \( \{ \neg P, \neg Q \} \vdash \neg (P \lor Q) \) in SD (proof below).

5. So \( \neg (P \lor Q) \) \( \in \Gamma^* \) (by 3, 4 and 6.4.9).

6. So it is not the case that \( \neg P \lor Q \in \Gamma^* \) (again by 6.4.11(a)).

So, if \( \neg P \lor Q \in \Gamma^* \), then \( P \in \Gamma^* \) or \( Q \in \Gamma^* \). Q.E.D.

Here is a proof of 4 (I think we did something very like this in class, but I do a derivation here anyway for completeness’ sake).

\[
\begin{array}{c|c}
1 & \neg P & A \\
2 & \neg Q & A \\
3 & P \lor Q & A/\neg I \\
4 & P & A/\lor E \\
5 & P & 4, R \\
6 & Q & A/\lor E \\
7 & \neg P & A/\neg E \\
8 & Q & 6, R \\
9 & \neg Q & 2, R \\
10 & P & 7-9, \neg E \\
11 & P & 3, 4-5, 6-10, \lor E \\
12 & \neg P & 1, R \\
13 & \neg (P \lor Q) & 3-12, \neg I \\
\end{array}
\]

Now to prove the other direction: if \( P \in \Gamma^* \) or \( Q \in \Gamma^* \), then \( \neg P \lor Q \in \Gamma^* \). Again, I do this by proving the contrapositive.

1. Suppose it is not the case that \( \neg P \lor Q \in \Gamma^* \).

2. Then \( \neg (P \lor Q) \in \Gamma^* \) (by 6.4.11(a)).

3. So \( \{ \neg (P \lor Q) \} \in \Gamma^* \).
4. Now, \( \{ \neg (P \lor Q) \} \vdash \neg P \) in SD, and \( \{ \neg (P \lor Q) \} \vdash \neg Q \) in SD (proof below).

5. So, \( \neg P \in \Gamma^* \) and \( \neg Q \in \Gamma^* \) (by 3, 4 and 6.4.9).

6. So it is not the case that either \( P \in \Gamma^* \) or \( Q \in \Gamma^* \) (by 6.4.11(a) again).

So, if \( P \in \Gamma^* \) or \( Q \in \Gamma^* \), then \( P \lor Q \in \Gamma^* \). Q.E.D.

Here is a proof of the first half of 4 — i.e., that \( \{ \neg (P \lor Q) \} \vdash \neg \neg P \).

<table>
<thead>
<tr>
<th></th>
<th>( \neg (P \lor Q) )</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( P )</td>
<td>A/\neg I</td>
</tr>
<tr>
<td>3</td>
<td>( P \lor Q )</td>
<td>2, \lor I</td>
</tr>
<tr>
<td>4</td>
<td>( \neg (P \lor Q) )</td>
<td>1, R</td>
</tr>
<tr>
<td>5</td>
<td>( \neg P )</td>
<td>2-4, \neg I</td>
</tr>
</tbody>
</table>

The proof of the other half of 4 is the same, except you replace the ‘\( P \)’s on lines 2 and 5 with ‘\( Q \)’s.

So, I’ve proven that if \( P \lor Q \in \Gamma^* \), then \( P \in \Gamma^* \) or \( Q \in \Gamma^* \), and I’ve proven that if \( P \in \Gamma^* \) or \( Q \in \Gamma^* \), then \( P \lor Q \in \Gamma^* \). It follows that \( P \lor Q \in \Gamma^* \) if and only if \( P \in \Gamma^* \) or \( Q \in \Gamma^* \). And that concludes the proof.

**Question 2**

We’re trying to prove Inductive Step, on p. 273 of TLB, for the case in which \( P \), a sentence containing \( k + 1 \) occurrences of connectives, has the form \( \neg Q \lor R \).

1. Suppose that every sentence of SL with \( k \) or fewer occurrences of connectives is such that it is true on \( A^* \) if and only if it is a member of \( \Gamma^* \) (i.e., suppose the antecedent of Inductive Step).

2. Now, \( \neg Q \lor R \) is true on \( A^* \) iff either \( Q \) is true on \( A^* \) or \( R \) is true on \( A^* \) (by definition of ‘\( \lor \)’).

3. And \( Q \) is true on \( A^* \) iff \( Q \in \Gamma^* \), and \( R \) is true on \( A^* \) iff \( R \in \Gamma^* \) (by 1, and the fact that \( Q, R \) both contain \( k \) or fewer occurrences of connectives).

4. So \( \neg Q \lor R \) is true on \( A^* \) iff either \( Q \in \Gamma^* \) or \( R \in \Gamma^* \) (from 2, 3).

5. So \( \neg Q \lor R \) is true on \( A^* \) if and only if \( \neg Q \lor R \in \Gamma^* \) (by 6.4.11(c) — i.e., the thing we just proved in Question 1).

So Inductive Step is true for the case in which \( P \) has the form \( \neg Q \lor R \). Q.E.D.
Question 3

The completeness proof for $SD$ will fail, as a proof for the completeness of $SD^*$, at the part where we try to prove 6.4.11(b) — i.e., the proof that \( \Gamma \models P \& Q \) if and only if both $P \in \Gamma^*$ and $Q \in \Gamma^*$ (where $\Gamma^*$ is a maximal consistent-in-$SD$ set of sentence of $SL$: $P$, $Q$ are sentence of SL) will not go through. In particular, the proof that if $\Gamma \models P \& Q$ then both $P \in \Gamma^*$ and $Q \in \Gamma^*$ will not go through. Note that the proof of that part of 6.4.11(b), on p. 272 of TLB, involves appealing to the Conjunction Elimination rule explicitly.

In fact, it will not, in general, be the case that a maximal consistent-in-$SD^*$ set is such that if $\Gamma \models P \& Q$ then both $P \in \Gamma^*$ and $Q \in \Gamma^*$ (though this is quite hard to prove, and I don’t do so here). There will, for example, be maximal consistent-in-$SD^*$ sets that are supersets of \{‘$A \& B$', ‘$\sim A$’\}.

Because the proof of 6.4.11(b) fails, the proof of what the book calls the ‘Consistency Lemma’ fails too; in particular, case 2 of the inductive step fails. Even more in particular, the part of case 2 in which we prove that if $\Gamma \models Q \& R$ is false on $A^*$ than it is not in $\Gamma^*$ will fail. That part of the proof relies on the part of 6.4.11(b) that fails without Conjunction Elimination. And you can see why: for a set that is a maximal consistent-in-$SD^*$ superset of \{‘$A \& B$', ‘$\sim A$’\}, ‘$A \& B$’ will be false on $A^*$, but it is in there anyway.