Logic I – Session 24

Completeness of PD
Last time

- We started to prove that PD is complete.
- We were working on proving that an ES-variant of any C-PD set is a subset of an MC-∃C-PD set \( \Gamma^* \).
- We set out a procedure for constructing \( \Gamma^* \), and proved that \( \Gamma^* \) was consistent.
- But there's one thing I need to correct...
Last time

Alex asked last time why we need to deal with evenly subscripted sets. In my answer, I misidentified the reason.

Our procedure for building $\Gamma^*$ included this clause:

If $\Gamma_i \cup \{P_i\}$ is C-PD, and $P_i$ is of the form $(\exists x)Q$:

$\Gamma_{i+1}$ is $\Gamma_i \cup \{P_i, Q(a/x)\}$, where $a$ is the alphabetically earliest constant not occurring in $P_i$ or any member of $\Gamma_i$

I asked, how do we know we can add $Q(a/x)$ to $\Gamma_k \cup \{P_i\}$?

The answer someone gave, and I agreed with, relied on the thought that finitely constants appear in any $\Gamma_i$ and $P_i$.

We were wrong.
Our main goal is to show that if $\Gamma \models P$ then $\Gamma \vdash \neg P$.

We do *not* assume initially that $\Gamma$ is finite --- that's why compactness is an interesting result.

So we do *not* assume that $\Gamma \cup \{\neg P\}$ is finite either!

So $\Gamma$ might contain infinitely many constants.
Now, in our construction of $\Gamma^*$, we sometimes need to add $Q(a/x)$ to $\Gamma_k$, where $a$ is a constant that isn’t in any sentences in the set.

How do we know this is possible?

The answer explains why we bother transforming $\Gamma \cup \{\neg P\}$ into an evenly subscripted set.

Let’s now turn back to proving that our construction of $\Gamma^*$ yields a set that’s $\text{MC-}\exists C\text{-PD}$.

We proved that $\Gamma^*$ is consistent. Let’s now prove that it’s maximally consistent.
An ES-variant of $\Gamma \cup \{\neg P\}$ is C-PD

Any ES-variant of $\Gamma \cup \{\neg P\} \subseteq$ a MC-∃C-PD set $\Gamma^*$ (11.4.4)

Goal: Show that $\Gamma^*$ is maximally consistent.

Suppose the contrary: There's a $P_k \not\in \Gamma^*$ s.t. $\Gamma^* \cup \{P_k\}$ is C-PD.

Since $P_k$ is a PL sentence, it occurs $k$th in our enumeration.

By the def. of our $\Gamma$-sequence, $\Gamma_{k+1} = \Gamma_k \cup \{P_k\}$ if that's C-PD.

$\Gamma_k \cup \{P_k\}$ is C-PD.

Since if $\{P_k\}$ were inconsistent with $\Gamma_k$, it would be inconsistent with every superset of $\Gamma_k$, e.g. $\Gamma^*$.

So $\Gamma_{k+1} = \Gamma_k \cup \{P_k\}$ (...perhaps plus a substitution instance)

But that means $P_k \in \Gamma_{k+1}$, so because $\Gamma_{k+1} \subseteq \Gamma^*$, $P_k \in \Gamma^*$.

Contradiction.

So there's no $P_k \not\in \Gamma^*$ s.t. $\Gamma^* \cup \{P_k\}$ is C-PD. I.e. $\Gamma^*$ is maximal.
An ES-variant of $\Gamma \cup \{\sim P\}$ is C-PD

Any ES-variant of $\Gamma \cup \{\sim P\} \subseteq$ a MC-∃C-PD set $\Gamma^*$ (11.4.4)

Finally, let’s show that $\Gamma^*$ is existentially complete.

I.e., that for each sentence in $\Gamma^*$ of the form $(\exists x)Q$, there’s a substitution instance of $(\exists x)Q$ in $\Gamma^*$.

So suppose $(\exists x)Q$ is in $\Gamma^*$.

$(\exists x)Q$ is in our enumeration, at some position $k$.

$\Gamma_k \cup \{P_k\}$, i.e. $\Gamma_k \cup \{(\exists x)Q\}$, is either C-PD or IC-PD.

If it were IC-PD, then since $(\exists x)Q$ is in $\Gamma^*$, $\Gamma^*$ would be IC-PD.

So $\Gamma_k \cup \{(\exists x)Q\}$ is C-PD.

But that means $\Gamma_{k+1}$ is $\Gamma_k \cup \{(\exists x)Q, Q(a/x)\}$, for some $a$.

So if $(\exists x)Q$ is in $\Gamma^*$, so is some substitution instance.
Completeness

\[ \Gamma \not\vdash P \]

\[ \square \]

An ES-variant of \( \Gamma \cup \{\neg P\} \) is C-PD

\[ \square \]

Any ES-variant of \( \Gamma \cup \{\neg P\} \) \( \subseteq \) a MC-∃C-PD set \( \Gamma^* \) (11.4.4)

\[ \square \]

If \( \Gamma^* \) is MC-∃C-PD then \( \Gamma^* \) is Q-C (11.4.8)

\[ \square \]

\[ \Gamma \cup \{\neg P\} \subseteq \text{a Q-C set} \; \Gamma^* \]

\[ \square \]

\[ \Gamma \cup \{\neg P\} \text{ is Q-C} \]

\[ \square \]

\[ \Gamma \not\models P \]
If $\Gamma^*$ is MC-$\exists C$-PD then $\Gamma^*$ is Q-C  (11.4.8)

Suppose $\Gamma^*$ is MC-PD and $\exists C$.

We’ll first prove some things about the membership of $\Gamma^*$, then we’ll give an interpretation on which every member is true.

This will show that $\Gamma^*$ is Q-C.
If $\Gamma^*$ is MC-$\exists$C-PD then $\Gamma^*$ is Q-C (11.4.8)

First, a helpful result: $\Gamma^* \vdash Q$ iff $Q \in \Gamma^*$.

Suppose $\Gamma^* \vdash Q$.

Suppose $Q \notin \Gamma^*$. Then since $\Gamma^*$ is MC, $\Gamma^* \cup \{Q\}$ is IC.

Then $\Gamma^* \cup \{Q\} \vdash R \& \sim R$.

So by $\sim$I, $\Gamma^* \vdash \sim Q$.

So $\Gamma^* \vdash Q \& \sim Q$, contradicting the fact that $\Gamma^*$ is MC.

So if $\Gamma^* \vdash Q$ then $Q \in \Gamma^*$.

Now suppose $Q \in \Gamma^*$.

Then since $\{Q\} \vdash Q$, and $\{Q\} \subseteq \Gamma^*$, $\Gamma^* \vdash Q$.

So if $Q \in \Gamma^*$, then $\Gamma^* \vdash Q$. 
If \( \Gamma^* \) is MC-∃C-PD then \( \Gamma^* \) is Q-C \((11.4.8)\)

Now let’s begin proving facts about the membership of \( \Gamma^* \).

- Suppose \( P \in \Gamma^* \).
  - Suppose \( \sim P \in \Gamma^* \).
  - Then \( \Gamma^* \) is IC-PD, contradicting what we’ve proved.
  - So \( \sim P \notin \Gamma^* \).
- So if \( P \in \Gamma^* \) then \( \sim P \notin \Gamma^* \). And contraposing: if \( \sim P \in \Gamma^* \) then \( P \notin \Gamma^* \).
If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C \hspace{1cm} (11.4.8)

Suppose $P \notin \Gamma^*$.

Since $\Gamma^*$ is MC, $\Gamma^* \cup \{P\}$ is IC-PD.

So $\Gamma^* \cup \{P\} \vdash Q \& \sim Q$.

But then $\Gamma^* \cup \{\sim \sim P\} \vdash Q \& \sim Q$. So $\Gamma^* \cup \{\sim \sim P\}$ is IC-PD.

So by $\sim E$, $\Gamma^* \vdash \sim P$.

We proved that if $\Gamma^* \vdash Q$, then $Q \in \Gamma^*$.

So since $\Gamma^* \vdash \sim P$, $\sim P \in \Gamma^*$.

So if $P \notin \Gamma^*$, then $\sim P \in \Gamma^*$. And contraposing: if $\sim P \notin \Gamma^*$ then $P \in \Gamma^*$. 
If \( \Gamma^* \) is MC-\( \exists C \)-PD then \( \Gamma^* \) is Q-C \((11.4.8)\)

So, we have: \( P \in \Gamma^* \) iff \( \neg P \notin \Gamma^* \) and, equivalently, \( P \notin \Gamma^* \) iff \( \neg P \in \Gamma^* \).

Similar results parallel our proofs about SD, e.g.:

- \( P \& Q \in \Gamma^* \) iff \( P \in \Gamma^* \) and \( Q \in \Gamma^* \).
- \( P \lor Q \in \Gamma^* \) iff \( P \in \Gamma^* \) or \( Q \in \Gamma^* \).
- \( P \supset Q \in \Gamma^* \) iff \( P \notin \Gamma^* \) or \( Q \in \Gamma^* \).
- \( P \equiv Q \in \Gamma^* \) iff \( P \in \Gamma^* \) and \( Q \in \Gamma^* \), or \( P \notin \Gamma^* \) and \( Q \notin \Gamma^* \).

New results will concern quantified sentences and their substitution instances.
If $\Gamma^*$ is MC-$\exists C$-PD then $\Gamma^*$ is Q-C \ (11.4.8)$

To prove: $(\exists x)P \in \Gamma^*$ iff for some constant $a$, $P(a/x) \in \Gamma^*$.

Suppose $(\exists x)P \in \Gamma^*$.

Since $\Gamma^*$ is $\exists C$, there's some $a$ such that $P(a/x) \in \Gamma^*$.

So if $(\exists x)P \in \Gamma^*$, then there's some $a$ such that $P(a/x) \in \Gamma^*$.

Now suppose for some constant $a$, $P(a/x) \in \Gamma^*$.

Since $\{P(a/x)\} \vdash (\exists x)P$ and $P(a/x) \in \Gamma^*$, we have $\Gamma^* \vdash (\exists x)P$.

We proved: $\Gamma^* \vdash Q$ iff $Q \in \Gamma^*$.

So $(\exists x)P \in \Gamma^*$.

So if there's some $a$ such that $P(a/x) \in \Gamma^*$, then $(\exists x)P \in \Gamma^*$.
If $\Gamma^*$ is MC-$\exists \mathcal{C}$-PD then $\Gamma^*$ is Q-$\mathcal{C}$ (11.4.8)

- So we've shown: $(\exists x)P \in \Gamma^*$ iff for some constant $a$, $P(a/x) \in \Gamma^*$.
- We can prove a similar result for universally quantified sentences.

To prove: $(\forall x)P \in \Gamma^*$ iff for every constant $a$, $P(a/x) \in \Gamma^*$.

- Suppose $(\forall x)P \in \Gamma^*$.
- For any $a$, since $\{ (\forall x)P \} \vdash P(a/x)$, $\Gamma^* \vdash P(a/x)$.
- We proved earlier that if $\Gamma^* \vdash Q$, then $Q \in \Gamma^*$.
- So $P(a/x) \in \Gamma^*$.
- So if $(\forall x)P \in \Gamma^*$, then for every constant $a$, $P(a/x) \in \Gamma^*$.
If $\Gamma^*$ is MC-∃C-PD then $\Gamma^*$ is Q-C (11.4.8)

- Now suppose: For every constant $a$, $P(a/x) \in \Gamma^*$.
- Suppose $(\forall x)P \notin \Gamma^*$.
- We proved that if $P \notin \Gamma^*$, then $\neg P \notin \Gamma^*$.
- So $\neg (\forall x)P \in \Gamma^*$.
- $\{\neg (\forall x)P\} \vdash (\exists x)\neg P$, so $\Gamma^* \vdash (\exists x)\neg P$.
- We proved earlier that if $\Gamma^* \vdash Q$, then $Q \in \Gamma^*$.
- So $(\exists x)\neg P \in \Gamma^*$.
- Since $\Gamma^*$ is ∃C, now there's some $a$ such that $\neg P(a/x) \in \Gamma^*$.
- Then $P(a/x) \notin \Gamma^*$! Contradicts our assumption.
- So it's not true that $(\forall x)P \notin \Gamma^*$. I.e., $(\forall x)P \in \Gamma^*$.
- So if, for every constant $a$, $P(a/x) \in \Gamma^*$, then $(\forall x)P \in \Gamma^*$. 
If $\Gamma^*$ is MC-$\exists C$-PD then $\Gamma^*$ is Q-C (11.4.8)

So here are our results about the membership of $\Gamma^*$.

$P \in \Gamma^*$ iff $\neg P \not\in \Gamma^*$

$P \land Q \in \Gamma^*$ iff $P \in \Gamma^*$ and $Q \in \Gamma^*$.

$P \lor Q \in \Gamma^*$ iff $P \in \Gamma^*$ or $Q \in \Gamma^*$.

$P \supset Q \in \Gamma^*$ iff $P \not\in \Gamma^*$ or $Q \in \Gamma^*$.

$P \equiv Q \in \Gamma^*$ iff $P \in \Gamma^*$ and $Q \in \Gamma^*$, or $P \not\in \Gamma^*$ and $Q \not\in \Gamma^*$.

$(\exists x)P \in \Gamma^*$ iff for some constant $a$, $P(a/x) \in \Gamma^*$.

$(\forall x)P \in \Gamma^*$ iff for every constant $a$, $P(a/x) \in \Gamma^*$.

This will allow us to prove that $\Gamma^*$ is Q-C by specifying an interpretation that makes all its members true.
If $\Gamma^*$ is MC-$\exists$C-PD then $\Gamma^*$ is Q-C (11.4.8)

Now recall, in proving the analogous result for SD, we just specified a truth-value assignment.

But interpretations for PL are more complicated, so we’ll need to do more.

In order to give an interpretation that works, let’s first remember that can alphabetically enumerate the individual constants of PL.

For any $n$, we have constants at positions $n$, $n+1$, $...$ in our enumeration.

Each constant can be thus be associated with a different unique positive integer.

- $a$ with 1, $b$ with 2, $...$ $v$ with 22, $a_1$ with 23, $...$ $v_4$ with 88, $...$
If $\Gamma^*$ is MC-$\exists C$-$PD$ then $\Gamma^*$ is Q-C (11.4.8)

- I* will be the following interpretation:
  - UD = \{n | n is a positive integer\}
  - For each sentence letter $P$, $I^*(P) = T$ iff $P \in \Gamma^*$. (Just like SD!)
  - For each i.c. $a$, $I^*(a)$ is the integer associated with $a$.
    - $a$: 1
    - $v4$: 88
    - ...
  - For each $n$-place predicate letter $A$, $I^*(A)$ includes all and only the $n$-tuples $<I^*(a_1), ..., I^*(a_n)>$ such that $Aa_1 ... a_n \in \Gamma^*$.

We'll show that every member of $\Gamma^*$ is true on I*, by proving that for any sentence $P$ of PL, $P$ is true on I* iff $P \in \Gamma^*$. 
If $\Gamma^*$ is MC-∃C-PD, then $\Gamma^*$ is Q-C \quad (11.4.8)$

Our proof will be by mathematical induction on the number of OLOs in sentences.

**Basis Clause:** For each sentence $P$ with zero OLOs,

$P$ is true on $I^*$ iff $P \in \Gamma^*$.

Proof: $P$ is either a sentence letter or atomic formula.

If $P$ is a sentence letter, $I^*(P) = T$ iff $P \in \Gamma^*$, by def. of $I^*$.

If $P$ is an atomic formula, then $P$ is of the form $Aa_1 \ldots a_n$.

- $<I^*(a_1), \ldots, I^*(a_n)> \in I^*(A)$ iff $Aa_1 \ldots a_n \in \Gamma^*$. [by def. of $I^*$]
- $Aa_1 \ldots a_n$ is true on $I^*$ iff $<I^*(a_1), \ldots, I^*(a_n)> \in I^*(A)$.
- So $Aa_1 \ldots a_n$ is true on $I^*$ iff $Aa_1 \ldots a_n \in \Gamma^*$.
- So if $P$ has zero OLOs, $P$ is true on $I^*$ iff $P \in \Gamma^*$. 

If $\Gamma^*$ is MC-$\exists\text{C-PD}$ then $\Gamma^*$ is Q-C (11.4.8)

**Inductive step:** If for each sentence $P$ with $k$ OLOs, $P$ is true on $I^*$ iff $P \in \Gamma^*$, then for each sentence $P$ with $k+1$ OLOs, $P$ is true on $I^*$ iff $P \in \Gamma^*$.

**Proof:** Suppose that for each sentence $P$ with $k$ OLOs, $P$ is true on $I^*$ iff $P \in \Gamma^*$.

Then $P$ has one of the following seven forms:

$\sim Q$, $Q \& R$, $Q \lor R$, $Q \supset R$, $Q \equiv R$, $(\forall x)Q$, or $(\exists x)Q$.

Thus there are 7 cases to test.

The cases where $P$ is a TF-compound are similar to cases we covered in discussing SD, so we'll only cover one.
If $\Gamma^*$ is MC-$\exists C$-PD then $\Gamma^*$ is Q-C (11.4.8)

Suppose $P$ is of the form $Q \lor R$.

To prove: $P$ is true on $I^*$ iff $P \in \Gamma^*$.

Suppose $P$ is true on $I^*$. Then either $Q$ or $R$ is true on $I^*$.

$Q$ and $R$ each have $k$ or fewer OLOs.

So by the inductive hypothesis, $Q$ is true on $I^*$ iff $Q \in \Gamma^*$, and $R$ is true on $I^*$ iff $R \in \Gamma^*$.

In each case, either $Q \in \Gamma^*$ or $R \in \Gamma^*$.

Earlier, we noted that $Q \lor R \in \Gamma^*$ iff $Q \in \Gamma^*$ or $R \in \Gamma^*$.

So $Q \lor R \in \Gamma^*$. I.e., $P \in \Gamma^*$.

So if $P$ is true on $I^*$, then $P \in \Gamma^*$.
If $\Gamma^*$ is MC-$\exists C$-$PD$ then $\Gamma^*$ is Q-$C$ \hfill (11.4.8)

Now suppose $P \in \Gamma^*$. I.e. $Q \lor R \in \Gamma^*$.

Now $Q \in \Gamma^*$ or $R \in \Gamma^*$, since $Q \lor R \in \Gamma^*$ iff $Q \in \Gamma^*$ or $R \in \Gamma^*$.

By the inductive hypothesis, $Q$ is true on $I^*$ iff $Q \in \Gamma^*$, and $R$ is true on $I^*$ iff $R \in \Gamma^*$.

So either $Q$ or $R$ is true on $I^*$.

If either $Q$ or $R$ is true on $I^*$, $Q \lor R$ is true on $I^*$.

So $Q \lor R$ is true on $I^*$. I.e. $P$ is true on $I^*$.

So if $P$ is of the form $Q \lor R$, then $P$ is true on $I^*$ iff $P \in \Gamma^*$. 
If \( \Gamma^* \) is MC-\( \exists \mathcal{C} \)-PD then \( \Gamma^* \) is Q-C \hfill (11.4.8)

We’ve supposed:
- For each sentence \( P \) with \( k \) OLOs, \( P \) is true on \( I^* \) iff \( P \in \Gamma^* \).
- And we’re trying to prove:
  - For each sentence \( P \) with \( k+1 \) OLOs, \( P \) is true on \( I^* \) iff \( P \in \Gamma^* \).
- We’re working by cases, for each form \( P \) could have.
- Next we’ll turn to cases where \( P \) is quantified.
If \( \Gamma^* \) is MC-∃C-PD then \( \Gamma^* \) is Q-C \quad (11.4.8)

- Suppose \( P \) is of the form \((\forall x)Q\).
- To prove: \( P \) is true on I* iff \( P \in \Gamma^* \).

  - Suppose \( P \) is true on I*. I.e., \((\forall x)Q\) is true on I*.
  - Then for every \( a \), \( Q(a/x) \) is true on I*.
  - For every \( a \), \( Q(a/x) \) has \( k \) or fewer OLOs, and hence by the inductive hypothesis, for every \( a \), \( Q(a/x) \in \Gamma^* \).
  - We earlier showed that \((\forall x)P \in \Gamma^* \) iff for every \( a \), \( P(a/x) \in \Gamma^* \).
  - So \((\forall x)Q \in \Gamma^* \). I.e. \( P \in \Gamma^* \).
  - So if \( P \) is true on I*, then \( P \in \Gamma^* \).
If \( \Gamma^* \) is MC-\( \exists C\)-PD then \( \Gamma^* \) is Q-C \quad (11.4.8)

Suppose \( P \) is false on \( I^* \). I.e. \( (\forall x)Q \) is false on \( I^* \).

\( (\forall x)Q \) is false on \( I^* \) iff for some ic \( a \), \( Q(a/x) \) is false on \( I^* \). (??)

(Heavy object: every object in the UD of \( I^* \) is named by some constant.)

So for some ic \( a \), \( Q(a/x) \) is false on \( I^* \).

For every ic \( a \), \( Q(a/x) \) has \( k \) or fewer OLOs, and hence by the inductive hyp., \( Q(a/x) \) is false on \( I^* \) iff \( Q(a/x) \notin \Gamma^* \).

So for some ic \( a \), \( Q(a/x) \notin \Gamma^* \).

Again, we know that \( (\forall x)Q \in \Gamma^* \) iff for every ic \( a \), \( Q(a/x) \in \Gamma^* \).

So \( (\forall x)Q \notin \Gamma^* \). I.e. \( P \notin \Gamma^* \).

So we've shown that if \( P \) is of the form \( (\forall x)Q \), \( P \) is true on \( I^* \) iff \( P \in \Gamma^* \).
If $\Gamma^*$ is MC-$\exists$C-PD then $\Gamma^*$ is Q-C \ (11.4.8)

I won’t go through the case for $P$ of the form $(\exists x)Q$, but the proof is similar. The completed mathematical induction would be:

**Basis Clause**: For each SL sentence $P$ with zero OLOs,
$P$ is true on $I^*$ iff $P \in \Gamma^*$.

**Inductive step**: If for each SL sentence $P$ with $k$ OLOs, $P$ is true on $I^*$ iff $P \in \Gamma^*$, then for each SL sentence $P$ with $k+1$ OLOs, $P$ is true on $I^*$ iff $P \in \Gamma^*$.

**Conclusion**: For any SL sentence $P$, $P$ is true on $I^*$ iff $P \in \Gamma^*$.

So there’s an interpretation that mem $\Gamma^*$ true. So $\Gamma^*$ is Q-C.
Completeness

Γ ⊭ P

An ES-variant of Γ ∪ \{¬P\} is C-PD

Any ES-variant of Γ ∪ \{¬P\} ⊆ a MC-∃C-PD set Γ* (11.4.4)

If Γ* is MC-∃C-PD then Γ* is Q-C (11.4.8)

Γ ∪ \{¬P\} ⊆ a Q-C set Γ*

Γ ∪ \{¬P\} is Q-C

Γ ⊭ P
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