1. (a) Bone is an orthotropic material, with the following elastic constants:

\[
\begin{align*}
E_1 &= 12.8 \text{ GPa} \\
E_2 &= 15.6 \text{ GPa} \\
E_3 &= 20.1 \text{ GPa} \\
G_{12} &= 4.7 \text{ GPa} \\
G_{13} &= 5.7 \text{ GPa} \\
G_{23} &= 6.7 \text{ GPa} \\
\nu_{12} &= \nu_{13} = \nu_{23} = 0.28
\end{align*}
\]

Calculate the elastic strain resulting from an applied stress field:

\[
\sigma = \begin{bmatrix} 10 & 0 & 3 \\ 0 & 5 & 4 \\ 3 & 4 & 20 \end{bmatrix} \text{ MPa.}
\]

(b) The Young’s modulus of a material is related to changes in the internal energy per atom, \(U_\omega\), and the entropy per atom, \(S_\omega\), with strain, \(\varepsilon\), according to:

\[
E = \frac{1}{\Omega} \left( \frac{d^2U_\omega}{de^2} - T \frac{d^2S_\omega}{de^2} \right)
\]

where \(\Omega\) is the atomic volume. Explain why the Young’s modulus of crystalline materials is controlled by the first term, while that for rubbers is controlled by the second term. Derive an expression for the Young’s modulus of crystalline materials in terms of the melting temperature and the atomic volume.

(c) Explain the origin of the bounds on Poisson’s ratio for an isotropic material:

\(-1 < \nu < 0.5\). Why do rubbers have values of Poisson’s ratio approaching 0.5?
2. (a) An aluminum alloy ($\sigma_y = 240$ MPa) component is subjected to the following stress state:

\[
\sigma = \begin{bmatrix}
200 & 40 & 0 \\
40 & 100 & 50 \\
0 & 50 & 0 \\
\end{bmatrix} \text{ MPa}
\]

Does the component yield according to the von Mises criterion?

(b) Why are covalently bonded materials intrinsically hard while metals are intrinsically soft?

(c) Why does the increase in yield strength from precipitation hardening depend on the surface energy of the precipitate while that from dispersion hardening depends on the shear modulus of the primary metal?

3. (a) What changes occur at the glass transition temperature that lead to viscoelastic behaviour in amorphous polymers?

(b) Describe two mechanisms of diffusion that give rise to diffusional flow creep behaviour in metals.
1. (a) \( E_1 = 12.8 \) GPa = 0
\( S_{11} = \frac{1}{E_1} = 7.81 \times 10^{-11} \) GPa

\( E_2 = 15.4 \) GPa = 0
\( S_{22} = \frac{1}{E_2} = 6.41 \times 10^{-11} \) GPa

\( E_3 = 20.1 \) GPa = 0
\( S_{33} = \frac{1}{E_3} = 4.98 \times 10^{-11} \) GPa

\( G_{12} = 4.4 \) GPa = 0
\( S_{46} = \frac{1}{2G_{12}} = 2.13 \times 10^{-10} \) GPa

\( G_{13} = 5.7 \) GPa = 0
\( S_{55} = \frac{1}{2G_{13}} = 1.75 \times 10^{-10} \) GPa

\( G_{23} = 6.7 \) GPa = 0
\( S_{44} = \frac{1}{2G_{23}} = 1.49 \times 10^{-10} \) GPa

\( \nu_{12} = \nu_{13} = \nu_{23} = 0.28 = 0 \)
\( \nu_{12} = - \frac{E_2}{E_1} = - \frac{S_{12}}{S_{11}} = 0 \)
\( S_{12} = - \frac{0.28}{12.8 \text{ GPa}} = - 2.19 \times 10^{-11} \)

\( S_{13} = - \frac{\nu_{13}}{E_1} = - \frac{0.28}{12.8} = - 2.19 \times 10^{-11} \)

\( S_{23} = - \frac{\nu_{23}}{E_2} = - \frac{0.28}{15.4} = - 1.79 \times 10^{-11} \)

\[
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
E_4 \\
E_5 \\
E_6
\end{bmatrix}
\begin{bmatrix}
7.81 & -2.19 & -2.19 \\
-2.19 & 6.41 & -1.79 \\
-2.19 & -1.79 & 4.98 \\
14.9 & & \\
17.5 & & \\
21.3 & &
\end{bmatrix}
\begin{bmatrix}
10 \\
5 \\
20 \\
40 \\
3 \\
0
\end{bmatrix}
= \begin{bmatrix}
2.34 \times 10^{-11} \\
-2.57 \times 10^{-11} \\
6.88 \times 10^{-11} \\
5.96 \times 10^{-11} \\
5.25 \times 10^{-11} \\
0
\end{bmatrix}
\]
4

(a) \( E_1 = 23.4 \times 10^{-5} \)  
\( E_2 = -25.7 \times 10^{-5} \)  
\( E_6 = 68.8 \times 10^{-5} \)  
\( E_4 = 2E_{23} = 5.96 \times 10^{-4} \)  
\( E_5 = 2E_{13} = 5.15 \times 10^{-4} \)  
\( E_6 = 2E_{16} = 0 \)

(b) \( E = \frac{1}{2} \left( \frac{d\mu e}{de} - T \frac{d\mu s}{de} \right) \)

Crystalline materials: uniaxial deformation atoms pulled apart or pushed together; relative positions don't change much; change in entropy small: \( d^2 S / de^2 \ll d\mu e / de \)

Rubbers: random C-C chains, occasionally cross-linked by covalent cross-links. Uniaxial deformation aligns chains, making them more ordered (chains can slide over one another without stretching C-C bonds along the length of the chain).

\( d^2 S / de^2 \ll d\mu e / de \)

Crystalline materials: \( E = f(\mu_a, S) \)

Dimensionless group: \( \frac{E_{\mu_a}}{\mu_a} = \text{constant} \)

\( \mu_a \times kT_m = D \)  
\( \frac{E_{\mu_a}}{kT_m} = \text{constant} \)

\( E = \frac{CkT_m}{\Omega} \)

(c) bounds on isotropic material

\( G = \frac{E}{2(1+v)} = D \)  \( v > -1 \)

\( K = \frac{E}{3(1-2v)} = D \)  \( v < \frac{1}{2} \)

Rubbers: large \( K \), small \( E \)

Hydrostatic pressure \( k \propto d^2 u / d e^2 \)  

Uniaxial stress \( E \times -\frac{d S}{d e} \)
2. (a) \( \sigma_y = 240 \text{ MPa} \)

\[
\sigma = \begin{bmatrix}
200 & 40 & 0 \\
40 & 100 & 50 \\
0 & 50 & 0
\end{bmatrix}
\]

Van Mises:

\[
\sigma_v = \sqrt{\frac{1}{2} \left( (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right) + 3\sigma_{12}^2 + 3\sigma_{13}^2 + 3\sigma_{23}^2}
\]

\[
= \sqrt{\frac{1}{2} \left( 100^2 + 4800 + 7500 \right) + 3(40)^2 + 3(50)^2}
\]

\[
= \sqrt{30,000 + 4800 + 7500}
\]

\[
= \sqrt{42,300}
\]

\[
= 206 \text{ MPa} < \sigma_y = 0 \text{ no yield}
\]

(b) Coval. bond - intra-ns hard

\[
\text{Unit} = \frac{1}{n^1} \text{ m x 6} = \text{ Unit falls rapidly with } r
\]

\[
\text{only nearest neighbors contribute to bonding; } z, \text{ no bonded neighbors per atom is small}
\]

\[
\text{energy per bond} = A_b = U_0 / (z/2) = 2U_0 / z
\]

\[
\frac{\text{E}_{\text{EL}}}{U_0} = \text{ constant (} = 3\text{)}
\]

\[
A_b = \frac{2}{3} \frac{\text{E}_{\text{EL}}}{z}
\]

\[
\text{work done by drill in breaking 1 bond} = c \text{(cm)}^3 \text{ per } \sigma \text{ x } \frac{T_a}{2}
\]

\[
\frac{c^2}{2} = \frac{2}{3} \frac{\text{E}_{\text{EL}}}{z} \Rightarrow \tau^0 = \frac{4}{3} \frac{\text{E}_{\text{EL}}}{z} \Rightarrow \text{small } \tau^0 \text{ large } \Rightarrow \text{ intra-ns hard}
\]
2(b) 
 _Intrinsic soft_ 
 
 \( \tau^0 = \frac{4}{3} \frac{E}{\varepsilon} \quad \tau^0 \text{ small (e.g. } \frac{\tau^0}{E} = 10^{-6}) \)

2(c) 
 _Precipitation hardening_

- Dislocation cuts through precipitate - new surface formed - requires surface energy 
- Energy balance: 

\[
\tau^0 \text{ppt} = 2\Gamma / L
\]

Dispersion hardening

- Dislocation pinned at hard dispersion obstacles
- Can't cut through
- Dislocation stretches - line tension = \( Gb^2 / 2 \) = change in strain energy when dislocation extends unit length
- Strain energy \( \varepsilon \) G, shear mod. of primary metal

Free balance 

\[
2\tau = \tau^0 b L
\]

\[
2 \frac{Gb^2}{2} = \tau^0 b L
\]

\[
\tau^0 = \frac{Gb}{L}
\]
3(a) \[ T_g \]
- sec. crosslink bonds betw chains melt
- free vol. increases with temp more rapidly
- motion of chains sliding across one another easier.

(b) Diffusional flow mechanisms
- vacancy diffusion

- grain boundary diffusion
  - grain boundary is a region of disorder
  - more open structure
  - high conductivity channel for diffusion
  - lower Q than for bulk/vacancy diffusion