Lecture 2. September 9, 2005

Homework. Problem Set 1 Part I: (f)–(h); Part II: Problems 3.

Practice Problems. Course Reader: 1C-2, 1C-3, 1C-4, 1D-3, 1D-5.

1. Tangent lines to graphs. For \( y = f(x) \), the equation of the secant line through \((x_0, f(x_0))\) and \((x_0 + \Delta x, f(x_0 + \Delta x))\) is,

\[
y = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}(x - x_0) + f(x_0).
\]

In the limit, the equation of the tangent line through \((x_0, f(x_0))\) is,

\[
y = f'(x_0)(x - x_0) + y_0.
\]

Example. For the parabola \( y = x^2 \), the derivative is,

\[
y'(x_0) = 2x_0.
\]

The equation of the tangent line is,

\[
y = 2x_0(x - x_0) = 2x_0x - x_0^2.
\]

For instance, the equation of the tangent line through \((2, 4)\) is,

\[
y = 4x - 4.
\]

Given a point \((x, y)\), what are all points \((x_0, x_0^2)\) on the parabola whose tangent line contains \((x, y)\)? To solve, consider \(x\) and \(y\) as constants and solve for \(x_0\). For instance, if \((x, y) = (1, -3)\), this gives,

\[
(-3) = 2x_0(1) - x_0^2,
\]

or,

\[
x_0^2 - 2x_0 - 3 = 0.
\]
Factoring \((x_0 - 3)(x_0 + 1)\), the solutions are \(x_0\) equals \(-1\) and \(x_0\) equals \(3\). The corresponding tangent lines are,

\[ y = -2x - 1, \]

and

\[ y = 6x - 9. \]

For general \((x, y)\), the solutions are,

\[ x_0 = \pm \sqrt{x^2 - y}. \]

2. Limits. Precise definition is on p. 791 of Appendix A.2. Intuitive definition: \(\lim_{x \to x_0} f(x)\) equals \(L\) if and only if all values of \(f(x)\) can be made arbitrarily close to \(L\) by choosing \(x\) sufficiently close to \(x_0\). One interpretation is the “microscope/laser illuminator” analogy: An observer focuses a microscopes field-of-view on a thin strip parallel to the x-axis centered on \(y = L\). The goal of the illuminator is to focus a laser-beam centered on \(x_0\) parallel to the y-axis (but with the line \(x = x_0\) deleted) so that only the portion of the graph in the field-of-view is illuminated. If for every magnification of the microscope, the illuminator can succeed, then the limit is defined and equals \(L\).

There is a beautiful Java applet on the webpage of Daniel J. Heath of Pacific Lutheran University,

http://www.plu.edu/~heathdj/java/calc1/Epsilon.html

If you use this, try \(a = -1\).

For left-hand limits, use a laser that illuminates only to the left of \(x_0\). For right-hand limits, use a laser that illuminates only to the right of \(x_0\).

3. Continuity. A function \(f(x)\) is continuous at \(x_0\) if \(f(x_0)\) is defined, \(\lim_{x \to x_0} f(x)\) is defined, and \(\lim_{x \to x_0} f(x)\) equals \(f(x_0)\). Also, \(f(x)\) is continuous on an interval if it is continuous at every point of the interval. The types of discontinuity are: removable discontinuity, jump discontinuity, infinite discontinuity and essential discontinuity.