Practice Exam 2 Solutions

Problem 1. Find
\[
\lim_{h \to 0} \frac{\int_0^{1+h} e^{t^2} \, dt - \int_0^1 e^{t^2} \, dt}{h(3+h^2)}.
\]

Solution First, using that the limit of a product is the product of limits, we get
\[
\lim_{h \to 0} \frac{\int_0^{1+h} e^{t^2} \, dt - \int_0^1 e^{t^2} \, dt}{h(3+h^2)} = \lim_{h \to 0} \frac{\int_0^{1+h} e^{t^2} \, dt - \int_0^1 e^{t^2} \, dt}{h} \lim_{h \to 0} \frac{1}{3+h^2}.
\]
Because \(\frac{1}{3+h^2}\) is a continuous function at \(h = 0\), the second limit is \(\frac{1}{3}\). Define
\[
g(x) = \int_0^x e^{t^2} \, dt.
\]
Then the first limit is
\[
g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = \frac{1}{3} e^{1^2} = e.
\]
By the fundamental theorem of calculus, \(g'(1) = e^{1^2} = e\). Multiplying the two limits together, our final answer is \(\frac{e}{3}\).

Problem 2. Find \((f^{-1})'(0)\) where \(f(x) = \int_0^x \cos(\sin(t)) \, dt\) is defined on \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

Solution First, we check that \(f\) is strictly increasing and continuous on \([-\frac{\pi}{2}, \frac{\pi}{2}]\). To show \(f\) is strictly increasing on \([-\frac{\pi}{2}, \frac{\pi}{2}]\), it is enough to show \(f'(x) > 0\) for all \(x \in (-\frac{\pi}{2}, \frac{\pi}{2})\). By the fundamental theorem of calculus, \(f'(x) = \cos(\sin(x))\). For \(x \in (-\frac{\pi}{2}, \frac{\pi}{2})\), we have \(\sin(x) \in (-1, 1)\), and for \(y \in (-1, 1)\), we have \(\cos y > 0\). Thus, \(f\) is strictly increasing on \([-\frac{\pi}{2}, \frac{\pi}{2}]\). Moreover, \(f\) is continuous on \([-\frac{\pi}{2}, \frac{\pi}{2}]\) by theorem 3.4 and differentiable on \((-\frac{\pi}{2}, \frac{\pi}{2})\) by the fundamental theorem of calculus (theorem 5.1). Then by theorem 6.7,
\[
(f^{-1})'(f(0)) = \frac{1}{f'(0)}.
\]
Since \(f(0) = 0\), and \(f'(0) = \cos(\sin(0)) = 1\) by the fundamental theorem of calculus (theorem 5.1), we deduce \((f^{-1})'(0) = 1\).
**Problem 3:** In each case below, assume \( f \) is continuous for all \( x \). Find \( f(2) \).

(a) 
\[
\int_0^x f(t)\,dt = x^2(1 + x).
\]

(b) 
\[
\int_0^{f(x)} t^2\,dt = x^2(1 + x).
\]

**Solution** (a) Differentiating both sides of the equality yields
\[
f(x) = 2x(1 + x) + x^2 = 3x^2 + 2x.
\]

To differentiate the left hand side, we used the fundamental theorem of calculus. To differentiate the right hand side, we used the product rule. Plugging in \( x = 2 \) yields
\[
f(2) = 16.
\]

(b) For this part, we integrate the left hand side to get
\[
\frac{f(x)^3}{3} = x^2(1 + x).
\]

Plugging in \( x = 2 \) and solving for \( f(2) \), we get \( f(2) = (36)^{\frac{1}{3}} \).

**Problem 4.** Give an example of a function \( f(x) \) defined on \([-1, 1]\) such that

- \( f \) is continuous and differentiable on \([-1, 1]\).
- \( f' \) is not continuous for at least one value \( x \in [-1, 1] \).

**Solution** Let
\[
f(x) = \begin{cases} 
x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0 \end{cases}.
\]

For \( x \neq 0 \), \( f(x) \) is a product of compositions of differentiable functions. Thus, \( f \) is differentiable for \( x \in [-1, 1], x \neq 0 \). Note
\[
0 \leq \left| \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} \right| \leq \left| \frac{h^2}{h} \right| = |h|.
\]

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Thus, using the squeezing principle (theorem 3.3), we deduce
\[
\lim_{h \to 0} \left| \frac{f(h) - f(0)}{h} \right| = 0.
\]
We conclude that \( f'(0) \) exists and equals zero. By a theorem from class, \( f \) is continuous on \([-1,1]\) because \( f \) is differentiable on \([-1,1]\). Next, we need to show that \( f' \) is discontinuous at \( x = 0 \). By the product rule and the chain rule, we have
\[
f'(x) = 2x \sin(1/x) - \cos(1/x) \text{ if } x \neq 0,
\]
and we know \( f'(0) = 0 \). Assume \( f' \) is continuous at \( x = 0 \). Then there must exist \( \delta > 0 \) such that \( |x| < \delta \) implies that \( |f'(x)| < \frac{1}{2} \). Choose \( x_0 = \frac{1}{2\pi n} < \delta \) with \( n \) a positive integer. Then
\[
f'(x_0) = 2 \frac{1}{2\pi n} \sin(2\pi n) - \cos(2\pi n) = 0 - 1 = -1.
\]
But, by assumption \( |f'(x_0)| < \frac{1}{2} \). This is a contradiction. Thus, \( f' \) is not continuous at \( x = 0 \).

**Problem 5.** Let \( f(x) \) be continuous on \([0,1]\), and assume \( f(0) = f(1) \). Show that for any \( n \in \mathbb{Z}^+ \), there exists at least one \( x \in [0,1] \) such that \( f(x) = f(x + \frac{1}{n}) \).

**Solution** Consider the continuous function \( g_n(x) = f(x) - f(x + \frac{1}{n}) \) on the interval \([0, \frac{n-1}{n}]\). Consider the set \( g_n(0), g_n(\frac{1}{n}), \ldots, g_n(\frac{n-1}{n}) \). If \( g_n(\frac{k}{n}) = 0 \) for some \( k \) then \( f(x + \frac{k}{n}) = f(x + \frac{k}{n} + \frac{1}{n}) \) and we are done. Hence, we may assume that \( g_n(\frac{k}{n}) \neq 0 \) for \( k = 0, 1, \ldots, n - 1 \). Note
\[
\sum_{k=0}^{n-1} g_n \left( \frac{k}{n} \right) = f(0) - f(1) = 0.
\]
If \( g_n(\frac{k}{n}) > 0 \) for \( k = 0, \ldots, n - 1 \), then the sum is positive, and if \( g_n(\frac{k}{n}) < 0 \) for \( k = 0, \ldots, n - 1 \), then the sum is negative. Since the sum is neither positive nor negative, there must be \( k_1 \) and \( k_2 \) such that \( g_n(\frac{k_1}{n}) > 0 \) and \( g_n(\frac{k_2}{n}) < 0 \). Putting \( y_1 = \min\{\frac{k_1}{n}, \frac{k_2}{n}\} \) and \( y_2 = \max\{\frac{k_1}{n}, \frac{k_2}{n}\} \), we note that \( g_n(y_1) \) and \( g_n(y_2) \) have opposite signs. Therefore, by the intermediate value theorem, there must be \( y \in (y_1, y_2) \) such that \( g_n(y) = 0 \). In particular,
\[
f(y) = f \left( y + \frac{1}{n} \right)
\]
as desired.