SECOND PRACTICE MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name:____________________
Signature:_________________
Recitation Time:_____________

There are 5 problems, and the total number of points is 100. Show all your work. Please make your work as clear and easy to follow as possible.

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1. (20pts) Find a recursive formula for a sequence of points \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), whose limit \((x_\infty, y_\infty)\), if it exists, is a point of intersection of the curves

\[
\begin{align*}
x^2 - y^2 &= 1 \\
x^2(x + 1) &= y^2.
\end{align*}
\]
2. (20pts) Suppose that \( F : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) is differentiable at \( P = (3, -2, 1) \) with derivative
\[
DF(3, -2, 1) = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & -3 \end{pmatrix}.
\]
Suppose that \( F(3, -2, 1) = (1, -3) \). Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be the function \( f(x, y, z) = \|F(x, y, z)\| \).

(i) Show that the function \( f(x, y, z) \) is differentiable at \( P \).

(ii) Find \( Df(3, -2, 1) \).

(iii) Find the directional derivative of \( f \) at \( P \) in the direction of \( \hat{u} = -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k} \).
3. (20pts) Let $F: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be a $C^1$ function. Suppose that

$$DF(3, 1, 0, -1) = \begin{pmatrix} 1 & 3 & 1 & 3 \\ -1 & 2 & -1 & -2 \end{pmatrix}.$$ 

(a) Show that there is an open subset $U \subset \mathbb{R}^2$ containing $(3, 1)$ and an open subset $V \subset \mathbb{R}^2$ containing $(0, -1)$ such that for all $(x, y) \in U$, the system of equations

$$F(x, y, z, w) = F(3, 1, 0, -1),$$

has the unique solution

$$(z, w) = (f_1(x, y), f_2(x, y)) \quad \text{with} \quad (z, w) \in V.$$

(b) Find the derivative $Df(3, 1)$.
4. (20pts) Let \( \mathbf{r}: I \rightarrow \mathbb{R}^3 \) be a regular smooth curve parametrised by arclength. Let \( a \in I \) and suppose that

\[
\mathbf{T}(a) = \frac{4}{9} \hat{i} - \frac{7}{9} \hat{j} - \frac{4}{9} \hat{k}, \quad \mathbf{B}(a) = \frac{1}{9} \hat{i} - \frac{4}{9} \hat{j} + \frac{8}{9} \hat{k}, \quad \frac{d\mathbf{N}}{ds}(a) = \hat{i} - 2\hat{j}.
\]

Find:
(i) the unit normal vector \( \mathbf{N}(a) \).

(ii) the curvature \( \kappa(a) \).

(iii) the torsion \( \tau(a) \).
5. (20pts) Let $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vector field given by $\vec{F}(x, y) = y\hat{i} + x\hat{j}$.

(i) Is $\vec{F}$ a gradient field (that is, is $\vec{F}$ conservative)? Why?

(ii) Is $\vec{F}$ incompressible?

(iii) Find a flow line that passes through the point $(1, 0)$.

(iv) Find a flow line that passes through the point $(a, b)$, where $a^2 > b^2$. 

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